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The Ramsey number of a long even cycle versus a star

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RAMSEY NUMBER $R(C_m, K_{1,n})$

DEFINITION

$R(C_m, K_{1,n})$ is the minimum number N so that each graph G on N vertices and the minimum degree at least $N - n$ contains a cycle C_m.

$R(C_m, K_{1,n})$ FOR SMALL n

OBSERVATION

If $n \leq m/2$ then $R(C_m, K_{1,n}) = m$.

Proof Obviously, $R(C_m, K_{1,n}) \geq m$.
Now, if $|V(G)| = m$, then

$$\delta(G) + \Delta(\overline{G}) = m - 1.$$

Therefore either $\delta(G) \geq \frac{m}{2}$ or $\Delta(\overline{G}) \geq \frac{m}{2}$. In the first case, by Dirac's Theorem, G is Hamiltonian (for $\delta(G) > \frac{m}{2}$, G is pancyclic), whereas in the second case clearly, as $n \leq \frac{m}{2}$, $K_{1,n} \subseteq \overline{G}$. So for n small enough $R(C_m, K_{1,n}) \leq m$. $\square$

Known results

Theorem Lawrence'73

If m is odd, then

$$R(C_m, K_{1,n}) = \begin{cases} m & \text{for} \quad m \geq 2n \\ 2n+1 & \text{for} \quad m \leq 2n-1\,. \end{cases}$$

Remark The lower bound in the second part follows from the complete bipartite graph $K_{n,n}$, i.e. the Turán graph $T_{n,2}$.

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Theorem Zhang, Broersma, Chen'16

If m is even, then

$$R(C_m, K_{1,n}) = \begin{cases} m & \text{for} \quad m \geq 2n \\ 2n & \text{for} \quad n < m \leq 2n \\ 2m-1 & \text{for} \quad 3n/4+1 \leq m \leq n\,. \end{cases}$$

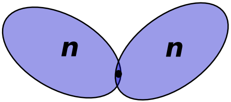
Extremal graphs

Theorem Zhang, Broersma, Chen'16

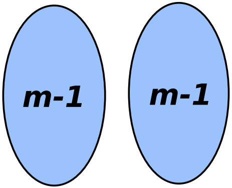
If m is even, then

$$R(C_m, K_{1,n}) = \begin{cases} m & \text{for} \quad m \geq 2n \\ 2n & \text{for} \quad n < m \leq 2n \\ 2m-1 & \text{for} \quad 3n/4 + 1 \leq m \leq n\,. \end{cases}$$

Extremal graphs:



$n < m \leq 2n$



$3n/4 + 1 \leq m \leq n$

Our main result

Theorem Łuczak, Polcyn, Zhang'20+

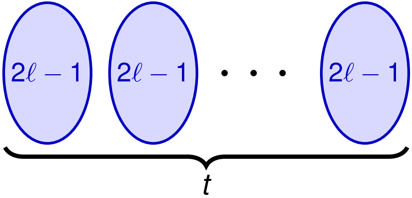
For every $t \geq 2$, $\ell \geq (19.1t)^9$, and n such that $(t-1)(2\ell-1) \leq n-1 < t(2\ell-1)$, we have

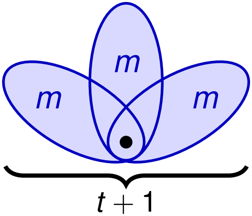
$$R(C_{2\ell}, K_{1,n}) = \max\{t(2\ell-1), n + \lfloor (n-1)/t \rfloor\} + 1\,.$$

Our main result: extremal graphs

$$(t-1)(2\ell-1) \le n-1 < t(2\ell-1)$$

As extremal graphs we choose the one with larger number of vertices:





$$m = \left\lfloor \frac{n-1}{t} \right\rfloor + 1$$

Sketch of the proof

We are to show that for $t \geq 2$ and ℓ large enough,

$$R(C_{2\ell}, K_{1,n}) \leq \max\{t(2\ell-1), n + \lfloor (n-1)/t \rfloor\} + 1,$$

where

$$(t-1)(2\ell-1) \leq n-1 < t(2\ell-1).$$

We let $G = (V, E)$ to be a $C_{2\ell}$-free graph on

$$|V| = \max\{t(2\ell-1), n + \lfloor (n-1)/t \rfloor\} + 1$$

vertices and such that $\Delta(\overline{G}) \leq n-1$.

A PLAN

STEP 1

The set of vertices of G contains a 'block-like' family of 2-connected subgraphs without vertices of very small degree.

STEP 2

Each subgraph in this family is small.

STEP 3

Then G has at most $\max\{t(2\ell-1), n+\lfloor(n-1)/t\rfloor\}+1$ vertices.

Step 1

Apply

Lemma

Let $n \geq k \geq 2$. For each graph G with n vertices and minimum degree $\delta(G) \geq n/k + k$, there exists an $s < k$ and a set of vertices $U \subset V(G)$, $|U| \leq s - 1$, such that $G - U$ is a union of s vertex-disjoint 2-connected graphs.

to G with $k = \frac{(t+1)^2+1}{t}$.

Thus, there exists $s \leq t + 2$ and a set of vertices $U \subset V$, $|U| \leq s - 1$, such that $G - U$ is a union of s vertex-disjoint, 2-connected graphs, $G_i' = (V_i', E_i')$.

Step 2

From theorems:

Theorem Williamson, 1977

Every graph $G = (V, E)$ on n vertices with $\delta(G) \geq n/2 + 1$ has the following property. For every $v, w \in V$ and every k such that $2 \leq k \leq n - 1$, G contains a path of length k which starts at v and ends at w. In particular, G is pancyclic. □

Theorem Voss and Zuluaga, 1977

Every 2-connected graph G on n vertices contains an even cycle C of length at least $\min\{2\delta(G), n - 1\}$. □

Step 2

Theorem Gould, Haxell, Scott, 2002

For every $a > 0$ there exists K such that each graph G on N vertices with $\delta(G) \geq aN$ contains a cycle of length r for every $r \in [4, \mathrm{ec}(G) - K]$, where $\mathrm{ec}(G)$ denotes the length of the longest even cycle contained in G.

follows

$$|V_i'| \leq 2\ell - 1, \quad \text{for} \quad i = 1, 2, \ldots, s.$$

STEP 3

For every $i = 1, 2, \ldots, s$, we define

$$U_i = \{u \in U : \deg_G(u, V_i') \geq 4t\}, \quad V_i = V_i' \cup U_i, \quad \text{and} \quad G_i = G[V_i].$$

Then the sets $V_1, V_2, \ldots, V_s$ satisfy

- $V = V_1 \cup V_2 \cup \cdots \cup V_s$,
- $|V_i| \leq 2\ell - 1$ for $i = 1, 2, \ldots, s$,
- $|V \setminus V_i| \leq n - 1$ for $i = 1, 2, \ldots, s$,
- $|V_1| + |V_2| + \cdots + |V_s| \leq |V| + s - 1$.

One can show that this implies

$$|V| \leq \max\{t(2\ell - 1), n + \lfloor (n-1)/t \rfloor\},$$

as required. □

Thank you for your attention!