Counting Dyck paths with colored hills: A bijective approach

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Abstract

It is well known that Dyck paths of semilength n are in bijection with 2-Motzkin paths of length $n-1$, i.e., both sets are enumerated by the n-th Catalan number C_n. Janjić [7] enumerated Dyck paths of semilength $n-1$ having colored hills with $m \in \{2, 3, 4\}$ colors. For $m = 2$, he showed that they are also enumerated by C_n, which implies that they are in bijection with 2-Motzkin paths of length $n-1$. For $m = 3$, he showed that they are enumerated by $\binom{2n-1}{n}$, which implies that they are in bijection with pairs of noncrossing paths [8] of length $n-1$.
In this work, we present new bijections between Dyck paths with colored hills and various other combinatorial objects, giving bijective proofs for the above results, as well as obtaining some new enumeration results.

Lattice path definitions

A (lattice) *path* of length $n \in \mathbb{N}^*$ is a finite sequence of points $(x_i, y_i)_{0 \leq i \leq n}$ in $\mathbb{Z}^2$, starting at the origin, i.e., $(x_0, y_0) = (0, 0)$. The vectors $(x_{i+1} - x_i, y_{i+1} - y_i)$, $0 \leq i \leq n-1$, are the *steps* of the path. The *length* of a path P, denoted by $|P|$, is the number of its steps. The height of the i-th point (x_i, y_i) of a path P, denoted by $h_i(P)$, is equal to y_i and $h(P) = h_n(P)$ is the height of the final point of P.
In this work, we are concerned with lattice paths having three kinds of steps: up steps $\mathrm{u} = (1, 1)$, down steps $\mathrm{d} = (1, -1)$ and horizontal steps $\mathrm{h} = (1, 0)$. The set of these paths is denoted by $\{\mathrm{u}, \mathrm{d}, \mathrm{h}\}^*$, since each such path can be identified by the sequence of its steps, i.e., a word in $\{\mathrm{u}, \mathrm{d}, \mathrm{h}\}^*$.

Lattice path definitions

Given $\tau, P \in \{\mathrm{u}, \mathrm{d}, \mathrm{h}\}^*$, we say that τ occurs in P whenever $P = R\tau Q$, for some $R, Q \in \{\mathrm{u}, \mathrm{d}, \mathrm{h}\}^*$. The height of this occurrence is equal to the minimum height of its points. The number of occurrences of τ in the path P is denoted by $|P|_\tau$. In particular, $|P|_{\mathrm{u}}, |P|_{\mathrm{d}}, |P|_{\mathrm{h}}$ denote the number of u's, d's and h's in P respectively.
A partial order $\leq$ is defined in the set $\{\mathrm{u}, \mathrm{d}\}^n$ of binary paths of length n, $n \in \mathbb{N}$, as follows: $P \leq Q$ whenever the binary path P lies weakly below the binary path Q, i.e., whenever $h_i(P) \leq h_i(Q)$, for all $0 \leq i \leq n$. A *pair of noncrossing paths* is a pair (P, Q) where $P \leq Q$.

Lattice path definitions

Definitions:

- Motzkin prefix: a path in $\{\mathrm{u}, \mathrm{d}, \mathrm{h}\}^*$ that stays weakly above the x-axis.
- Motzkin path: a Motzkin prefix that ends on the x-axis.
- Dyck prefix: a Motzkin prefix with no horizontal steps (also called a ballot path).
- Dyck path: a Dyck prefix that ends on the x-axis.
- A *hill* of a path is an occurrence of ud at height 0 (the starting point of the u step has zero y-coordinate).

Lattice path definitions - Colored paths

By coloring each horizontal step of a Motzkin prefix with one out of $m \in \mathbb{N}^*$ possible colors, we obtain an *m-Motzkin prefix*. We denote these colors by the integers $1, 2, \dots, m$ and the corresponding colored horizontal steps by $h_1, h_2, \dots, h_m$.
Similarly, we can color the hills of a Dyck path to obtain a *Dyck path with m-colored hills*.
We denote these colored hills by $H_1, H_2, \dots, H_m$.

Lattice path definitions - Notation

Notation:

- ε: the empty path, i.e., the path of length 0.
- $\mathcal{MP}_n^{(m)}$: The set of m-Motzkin prefixes of length n. $\mathcal{MP}^{(m)} = \bigcup_{n\geq 0} \mathcal{MP}_n^{(m)}$.
- $\mathcal{MP}_n^{(m)}(h)$: The set of m-Motzkin prefixes of length n ending at height h.
- $\mathcal{M}_n^{(m)}$: The set of m-Motzkin paths of length n. $\mathcal{M}^{(m)} = \bigcup_{n\geq 0} \mathcal{M}_n^{(m)}$.
- $\mathcal{DP}_n^{(m)}$: The set of Dyck prefixes of length n with m-colored hills. $\mathcal{DP}^{(m)} = \bigcup_{n\geq 0} \mathcal{DP}_n^{(m)}$.
- $\mathcal{DP}_n(h)$: The set of Dyck prefixes of length n ending at height h.

Lattice path definitions - Notation

Notation:

- $\mathcal{D}_n^{(m)}$: The set of Dyck paths of length $2n$ with m-colored hills. $\mathcal{D}^{(m)} = \bigcup_{n\geq 0} \mathcal{D}_n^{(m)}$.
- $\mathcal{D}_n^{(0)}$: The set of Dyck paths of length $2n$ with no hills (also called hill-free Dyck paths or Fine paths). $\mathcal{D}^{(0)} = \bigcup_{n\geq 0} \mathcal{D}_n^{(0)}$.
- $\mathcal{D}_n = \mathcal{D}_n^{(1)}$, $\mathcal{D} = \mathcal{D}^{(1)}$, $\mathcal{D}^* = \mathcal{D} \setminus \{\varepsilon\}$.
- $\mathcal{W}_n(h)$: The set of pairs of noncrossing binary paths of length n ending $2h$ units apart. $\mathcal{W}_n = \bigcup_{h\geq 0} \mathcal{W}_n(h)$, $\mathcal{W} = \bigcup_{n\geq 0} \mathcal{W}_n$, $\mathcal{W}(h) = \bigcup_{n\geq 0} \mathcal{W}_n(h)$.

Known results

Combining various enumeration results from the literature, we have the following equalities:

$$|\mathcal{W}_n(0)| = |\mathcal{D}_{n+1}| = |\mathcal{M}_n^{(2)}| = |\mathcal{D}_n^{(2)}| = C_{n+1},$$

$$|\mathcal{DP}_{2n+1}| = |\mathcal{W}_n| = |\mathcal{MP}_n^{(2)}| = |\mathcal{D}_n^{(3)}| = \binom{2n+1}{n}.$$

There exist known bijections between the sets $\mathcal{W}_n(0)$ and $\mathcal{D}_{n+1}$ (see Deutsch and Shapiro[6], Manes et al. [8]), as well as between the sets $\mathcal{D}_{n+1}$ and $\mathcal{M}_n^{(2)}$ (see Callan [3], Delest and Viennot [5]).

Moreover, a bijection between $\mathcal{W}_n(0)$ and $\mathcal{M}_n^{(2)}$ is given by Deutsch and Shapiro [6], and a bijection between $\mathcal{W}_n$ and $\mathcal{DP}_{2n+1}$ is given by Manes et al. [8].

Known results

Recently, Janjić [7] proved, using recurrence relations, the following enumeration results:

i) $|\mathcal{D}_n^{(2)}| = C_{n+1}$, where $C_n = \binom{2n}{n}/(n+1)$ is the n-th Catalan number (seq. A000108 in the OEIS [9]). As Janjić notes, this interpretation of the Catalan numbers does not exist in Stanley's book "Catalan numbers" [10].

ii) $|\mathcal{D}_n^{(3)}| = \binom{2n+1}{n}$ (seq. A001700 in the OEIS). Again, no bijective proof is known for this result.

New results

The purpose of this work is to provide bijective proofs for (i) and (ii). We introduce a simple new bijection $\phi : \mathcal{D}^{(2)} \to \mathcal{D}^*$, which proves (i). Moreover, we introduce another, more complicated bijection $\varphi_2 : \mathcal{D}^{(2)} \to \mathcal{M}^{(2)}$, also proving (i), and we study some of its properties, obtaining some known and some new results.

In order to define φ_2, we first introduce a new bijection $\varphi_1 : \mathcal{D}^* \to \mathcal{M}^{(2)}$ between Dyck paths and 2-Motzkin paths, which gives an alternative bijective proof of the equality $|\mathcal{D}_{n+1}| = |\mathcal{M}_n^{(2)}|$.

We also introduce two new bijections $\varphi_3 : \mathcal{D}^{(3)} \to \mathcal{MP}^{(2)}$ and $\chi : \mathcal{W} \to \mathcal{MP}^{(2)}$. Then $\chi^{-1} \circ \varphi_3$ is a bijection that proves (ii).

From Dyck paths with 2-colored hills to Dyck paths

The mapping $\phi : \mathcal{D}^{(2)} \to \mathcal{D}^*$ has a simple non-recursive description; for every $\alpha \in \mathcal{D}^{(2)}$, the path $\phi(\alpha)$ is constructed in two steps as follows:

1. Transform each H_2 (hill with color 2) of α into a du (a valley reaching height -1).
2. Finally, add a u step at the beginning and a d step at the end of the path.

Obviously the resulting path $\phi(\alpha)$ is a non-empty Dyck path such that $|\phi(\alpha)|_{\mathrm{u}} = |\alpha|_{\mathrm{u}} + 1$. Moreover, the procedure is clearly reversible, so that we have a bijection, showing that $|\mathcal{D}_n^{(2)}| = |\mathcal{D}_{n+1}| = C_{n+1}$, for all $n \in \mathbb{N}$.

From Dyck paths to 2-Motzkin paths

As noted before, there exists a folklore bijection $\psi : \mathcal{M}^{(2)} \rightarrow \mathcal{D}^*$, introduced by Delest and Viennot [6], which has a straightforward description: Given a 2-Motzkin path α, the Dyck path $\phi(\alpha)$ is constructed by replacing in α each u by uu, each d by dd, each h_1 by ud, each h_2 by du and then by adding a u step at the beginning and a d step at the end of the path.

From Dyck paths to 2-Motzkin paths

The bijection $\varphi_1 : \mathcal{D}^* \to \mathcal{M}^{(2)}$ that we introduce here, mapping non-empty Dyck paths of length $2n$ to 2-Motzkin paths of length $n+1$, is quite different from ψ^{-1}. Its definition is recursive and it is based on the decompositions of the two sets.

A path $\alpha \in \mathcal{D}^*$ is decomposed as

$$\alpha = \mathrm{ud} \quad \text{or } \alpha = \mathrm{u}\beta\mathrm{d} \quad \text{or} \quad \alpha = \mathrm{ud}\beta \quad \text{or} \quad \alpha = \mathrm{u}\beta\mathrm{d}\gamma, \quad \beta, \gamma \in \mathcal{D}^*.$$

On the other hand, a path $\alpha \in \mathcal{M}^{(2)}$ is decomposed as

$$\alpha = \varepsilon \quad \text{or} \quad \alpha = \mathrm{h}_1\beta \quad \text{or} \quad \alpha = \mathrm{h}_2\beta \quad \text{or} \quad \alpha = \mathrm{u}\beta\mathrm{d}\gamma, \quad \beta, \gamma \in \mathcal{M}^{(2)}.$$

Then, φ_1 is defined recursively as

$$\varphi_1(\mathrm{ud}) = \varepsilon, \quad \varphi_1(\mathrm{u}\beta\mathrm{d}) = \mathrm{h}_1\varphi_1(\beta), \quad \varphi_1(\mathrm{ud}\beta) = \mathrm{h}_2\varphi_1(\beta),$$
$$\varphi_1(\mathrm{u}\beta\mathrm{d}\gamma) = \mathrm{u}\varphi_1(\beta)\mathrm{d}\varphi_1(\gamma), \qquad \beta, \gamma \in \mathcal{D}^*,$$

as depicted in the next figure.

From Dyck paths to 2-Motzkin paths

$$\alpha \in \mathcal{D}^* \quad \longleftrightarrow \quad \varphi_1(\alpha) \in \mathcal{M}^{(2)}$$

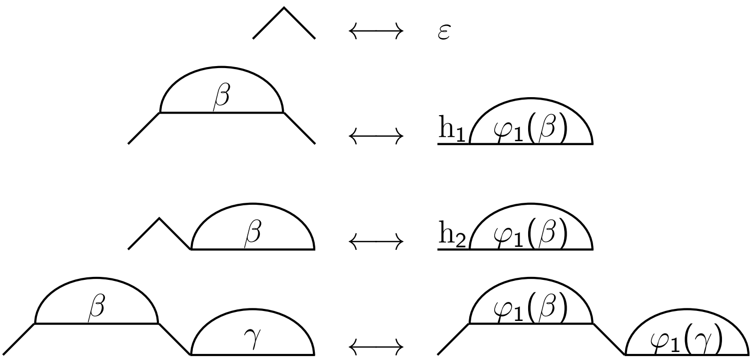

Figure: The bijection $\varphi_1 : \mathcal{D}^* \to \mathcal{M}^{(2)}$.

From Dyck paths to 2-Motzkin paths

Using the recursive definition, it is easy to prove inductively that φ_1 is a bijection having the following properties:

- $|\varphi_1(\alpha)| = |\alpha|_{\mathrm{u}} + 1$,
- $|\varphi_1(\alpha)|_{\mathrm{h}_2} = |\alpha|_{\mathrm{udu}}$,
- $|\varphi_1(\alpha)|_{\mathrm{d}} = |\alpha|_{\mathrm{ddu}}$,
- $|\varphi_1(\alpha)|_{\mathrm{u}} + |\varphi_1(\alpha)|_{\mathrm{h}_2} = |\alpha|_{\mathrm{uu}} = |\alpha|_{\mathrm{dd}}$.

Remark:

According to the second property, by restricting φ_1 to the set of udu-free Dyck paths of length $2n+2$, we get a bijection onto Motzkin paths of length n, reobtaining the known result (e.g., see Callan [3]) that the former are counted by the Motzkin numbers M_n (seq. A001006 in the OEIS).

From Dyck paths with 2-colored hills to 2-Motzkin paths

The mapping $\varphi_2 : \mathcal{D}^{(2)} \to \mathcal{M}^{(2)}$ maps Dyck paths of length $2n$ with 2-colored hills to 2-Motzkin paths of length n. Its definition is based on the decompositions of the two sets.
A nonempty path $\alpha \in \mathcal{D}^{(2)}$ is decomposed with respect to its first hill as

$$\alpha = \mathrm{u}\alpha_1\mathrm{d}\cdots\mathrm{u}\alpha_k\mathrm{d} \quad \text{or} \quad \alpha = \beta H_1 \gamma \quad \text{or} \quad \alpha = \beta H_2 \gamma,$$

where $\beta \in \mathcal{D}^{(0)}, \gamma \in \mathcal{D}^{(2)}, \alpha_i \in \mathcal{D}^*, i \in [k], k \in \mathbb{N}^*$.
Note that the first equality corresponds to the case where α is hill-free, whereas the second and third equalities correspond to the cases where α has a hill.

From Dyck paths with 2-colored hills to 2-Motzkin paths

On the other hand, a nonempty path $\alpha \in \mathcal{M}^{(2)}$ is decomposed analogously, with respect to its first horizontal step at height 0, as

$$\alpha = \mathrm{u}\alpha_1\mathrm{d}\cdots\mathrm{u}\alpha_k\mathrm{d} \quad \text{or} \quad \alpha = \beta\mathrm{h}_1\gamma \quad \text{or} \quad \alpha = \beta\mathrm{h}_2\gamma,$$

where $\beta \in \overline{\mathcal{M}}^{(2)}, \gamma, \alpha_i \in \mathcal{M}^{(2)}, i \in [k], k \in \mathbb{N}^*$ and $\overline{\mathcal{M}}^{(m)}$ is the set of m-Motzkin paths with no horizontal steps at height 0.

Then, φ_2 is defined recursively as

$$\varphi_2(\varepsilon) = \varepsilon, \quad \varphi_2(\mathrm{u}\alpha_1\mathrm{d}\cdots\mathrm{u}\alpha_k\mathrm{d}) = \mathrm{u}\varphi_1(\alpha_1)\mathrm{d}\cdots\mathrm{u}\varphi_1(\alpha_k)\mathrm{d},$$

$$\varphi_2(\beta H_i\gamma) = \varphi_2(\beta)\mathrm{h}_i\varphi_2(\gamma), \qquad i \in \{1, 2\},$$

where $a_1, \ldots, a_k \in \mathcal{D}^*$, $\beta \in \mathcal{D}^{(0)}$, $\gamma \in \mathcal{D}^{(2)}$.

From Dyck paths with 2-colored hills to 2-Motzkin paths

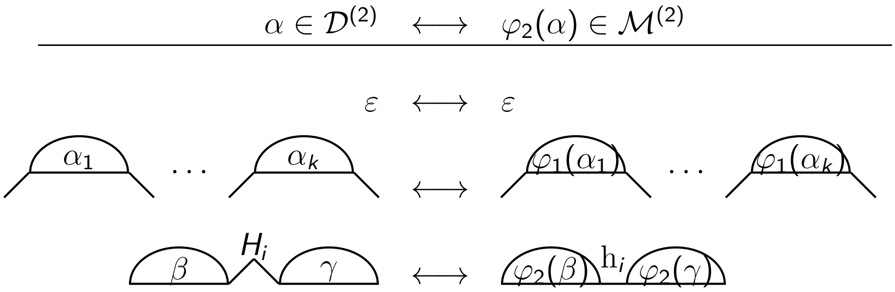

Figure: The bijection $\varphi_2 : \mathcal{D}^{(2)} \to \mathcal{M}^{(2)}$.

From Dyck paths with 2-colored hills to 2-Motzkin paths

It is easy to prove inductively that φ_2 is a bijection having the following properties:

- $|\varphi_2(\alpha)| = |\alpha|_{\mathrm{u}}$.
- #hills in α = #horizontal steps at height 0 in $\varphi_2(\alpha)$.
- The restriction of φ_2 on $\mathcal{D}^{(0)}$ is a bijection onto $\overline{\mathcal{M}}^{(2)}$. This verifies the well-known result that $|\mathcal{D}_n^{(0)}| = |\overline{\mathcal{M}}_n^{(2)}| = \mathbb{F}_{n+1}$, where $\mathbb{F}_n$ is the n-th Fine number (seq. A000957 in the OEIS).

From Dyck paths with 2-colored hills to 2-Motzkin paths

Remarks:

- Restricting φ_2 on hill-free Dyck paths with no udu's, we get a bijection onto $\overline{\mathcal{M}}^{(1)}$, i.e., Motzkin paths with no horizontal steps at height 0. It is known that $|\overline{\mathcal{M}}_n^{(1)}| = R_n$ (seq. A005043 in the OEIS). Thus, we have a new interpretation of the numbers R_n; they count hill-free Dyck paths of length $2n$ with no udu's, or equivalently udu-free Dyck paths of length $2n$ ending with dd.
- Since udu-free Dyck paths of length $2n+2$ are counted by the Motzkin numbers M_n (seq. A001006 in the OEIS), and they end with either dd or ud, we obtain a combinatorial interpretation of the known relation $M_n = R_{n+1} + R_n$.

From Dyck paths with 3-colored hills to 2-Motzkin prefixes

The mapping $\varphi_3 : \mathcal{D}^{(3)} \to \mathcal{MP}^{(2)}$ maps Dyck paths of length $2n$ with 3-colored hills to 2-Motzkin prefixes of length n. Its definition is based on the decompositions of the two sets. A path $\alpha \in \mathcal{D}^{(3)} \setminus \mathcal{D}^{(2)}$ is decomposed with respect to its first hill with color 3 as

$$\alpha = \beta H_3 \gamma, \qquad \beta \in \mathcal{D}^{(2)}, \gamma \in \mathcal{D}^{(3)}.$$

Analogously a 2-Motzkin prefix $\alpha \in \mathcal{MP}^{(2)} \setminus \mathcal{M}^{(2)}$ is decomposed with respect to its last u step reaching height 1 as

$$\alpha = \beta \mathrm{u} \gamma, \qquad \beta \in \mathcal{M}^{(2)}, \gamma \in \mathcal{MP}^{(2)}.$$

From Dyck paths with 3-colored hills to 2-Motzkin prefixes

Then, φ_3 is defined recursively, using φ_2, as follows:

$$\varphi_3(\alpha) = \varphi_2(\alpha) \text{ and } \varphi_3(\beta H_3 \gamma) = \varphi_2(\beta)\mathrm{u}\varphi_3(\gamma),$$

where $\alpha, \beta \in \mathcal{D}^{(2)}, \gamma \in \mathcal{D}^{(3)}$.
Using the recursive definition, it is easy to prove inductively that φ_3 is a bijection having the following properties:

- $|\varphi_3(\alpha)| = |\alpha|_{\mathrm{u}}$.
- $\varphi_3(\alpha) \in \mathcal{MP}^2(k) \Leftrightarrow |\alpha|_{H_3} = k$, i.e., the number of H_3's in α equals the ending height of $\varphi_3(\alpha)$.
- $\varphi_3(\alpha)$ has no horizontal steps at height 0 iff either α is hill-free or its first hill has color 3.

From Dyck paths with 3-colored hills to 2-Motzkin prefixes

Remarks:

- If we choose to map each H_3 to an h_3 instead of a u, then we get a bijection onto 3-Motzkin paths where the h_3's occur only at height 0 (see a comment by Deutsch in seq A001700 in the OEIS).
- According to the third property stated before, the number of paths in $\mathcal{D}_n^{(3)}$ with exactly k hills of color 3 is equal to $|\mathcal{MP}^{(2)}(k)| = \dfrac{k+1}{n+1}\dbinom{2n+2}{n-k}$ (seq. A039598 in the OEIS).

Pairs of noncrossing paths

A pair $(P, Q) \in \mathcal{W}(0) \setminus \{(\varepsilon, \varepsilon)\}$ is decomposed as

$$(\mathrm{u}P', \mathrm{u}Q') \quad \text{or} \quad (\mathrm{d}P', \mathrm{d}Q') \quad \text{or} \quad (\mathrm{d}P'\mathrm{u}P'', \mathrm{u}Q'\mathrm{d}Q''),$$

where $(P', Q'), (P'', Q'') \in \mathcal{W}(0)$.
The first two cases occur whenever P and Q start with a joint step, so that their remaining parts P' and Q' clearly form a pair in $\mathcal{W}(0)$. The third case occurs whenever the initial step is not a joint step, so that P must start with a d and Q with a u (since $P \leq Q$) and the paths must meet again with an upstep for P and a downstep for Q. If $\mathrm{d}P'\mathrm{u}$ and $\mathrm{u}Q'\mathrm{d}$ are their initial parts until their first common point, then $\mathrm{d}P'$ and $\mathrm{u}Q'$ have no common points (other than $(0, 0)$) and the distance between their ending points is 2, so that $(P', Q') \in \mathcal{W}(0)$. Obviously, the remaining parts P'' and Q'' also form a pair in $\mathcal{W}(0)$.

Pairs of noncrossing paths

Furthermore, a pair $(P, Q) \in \mathcal{W} \setminus \mathcal{W}(0)$ is decomposed uniquely with respect to the last common point as

$$(P'\mathrm{d}P''', Q'\mathrm{u}Q'''), \qquad \text{where } (P', Q') \in \mathcal{W}(0), (P''', Q''') \in \mathcal{W}.$$

Here, P' and Q' are the initial parts of P and Q until their last common point (these parts are empty iff no common point exists). The remaining parts dP''' and uQ''' have no common point (other than $(0,0)$) so that $(P''', Q''') \in \mathcal{W}$.
Then, the bijection $\chi : \mathcal{W} \to \mathcal{MP}^{(2)}$ is defined recursively, based on the decompositions of the two sets, as shown in the next figure.

Pairs of noncrossing paths

$$(P,Q) \in \mathcal{W} \quad \longleftrightarrow \quad \chi(P,Q) \in \mathcal{MP}^{(2)}$$

$$(\varepsilon,\varepsilon) \quad \longleftrightarrow \quad \varepsilon$$

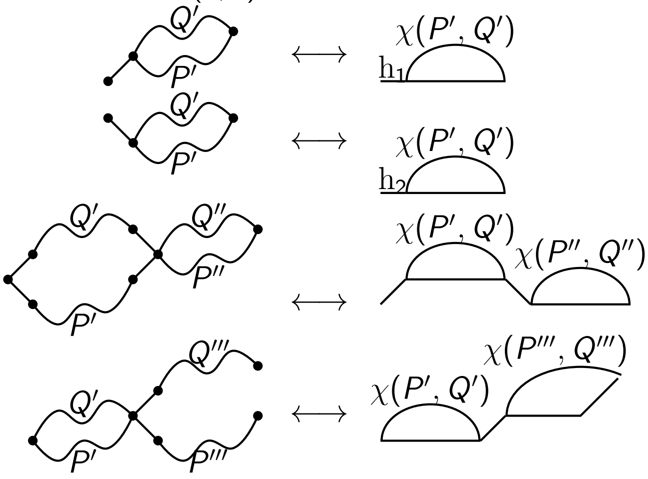

Figure: The bijection $\chi : \mathcal{W} \to \mathcal{MP}^{(2)}$.

Pairs of noncrossing paths

Using the recursive definition of χ, it is easy to prove inductively that χ is a bijection having the following properties:

- $|\chi(P, Q)| = |P|$.
- The restriction of χ on $\mathcal{W}(0)$ is a bijection onto $\mathcal{M}^{(2)}$, showing that $|\mathcal{W}_n(0)| = C_{n+1}$.
- The restriction of χ on $\mathcal{W}(h)$ is a bijection onto $\mathcal{MP}^{(2)}(h)$.
- Joint u's of (P, Q) correspond to h_1's at height 0 of $\chi(P, Q)$.
- Joint d's of (P, Q) correspond to h_2's at height 0 of $\chi(P, Q)$.
- χ maps (u,u)'s to h_1's, (d,d)'s to h_2's, (d,u)'s to u's and (u,d)'s to d's.

Pairs of noncrossing paths

Remarks

- The last property implies that χ has a simple non-recursive description. Read the pairs of steps of the pair (P, Q) and transform (u, u)'s into h_1's, (d, d)'s into h_2's, (d, u)'s into u's and (u, d)'s into d's.
- The restriction on $\mathcal{W}(0)$ coincides with the bijection given by Deutsch and Shapiro [6].
- The mapping $\chi^{-1} \circ \varphi_3$ is a bijection verifying that $|\mathcal{D}_n^{(3)}| = |\mathcal{W}_n| = \binom{2n+1}{n}$.

Pairs of noncrossing paths

Remark

Taking all previous facts into account, we can give a bijection $\omega : \mathcal{W} \to \mathcal{D}^{(3)}$ with a simple description consisting of three steps:

1. Each (u, u) is replaced by ud, each (d, d) by du, each (d, u) by uu and each (u, d) by dd.
2. Then, the uu's starting with an unmatched u at even height are replaced by H_3's.
3. Finally, each du below zero level is turned into an H_2.

The restriction of ω on $\mathcal{W}(0)$ is the bijection $\phi^{-1} \circ \psi \circ \chi : \mathcal{W}(0) \to \mathcal{D}^{(2)}$, which is described by ignoring step 2.

Generating functions

The invert transform (IT) (see Bernstein and Sloane [1], Cameron [4]), denoted here by $\mathcal{I}$, of a generating function $A(x) = \sum_{n\geq 1} a_n x^n$ is the generating function

$$\mathcal{I}A(x) := \frac{A(x)}{1 - A(x)} = \sum_{k\geq 1} A(x)^k.$$

It follows inductively that the m-th IT of A is equal to

$$\mathcal{I}^m A(x) = \frac{A(x)}{1 - mA(x)} = \sum_{k\geq 1} m^k A(x)^k, \qquad m \in \mathbb{N}^*.$$

We define $\mathcal{I}^m(a_n) := [x^n]\mathcal{I}^m A(x)$.

Generating functions

There is a straightforward combinatorial interpretation of the IT in terms of words, as explained by Birmajer et al. in [2]: If a_n is the number of words of length $n-1$ over the alphabet $\mathcal{L} = \{0, 1, 2, \ldots, b-1\}$, $b \in \mathbb{N}^*$, then $\mathcal{I}(a_n)$ is the number of words of length $n-1$ over $\mathcal{L} \cup \{b\}$. This is obvious, since $(\mathcal{L} \cup \{b\})^{n-1}$ is the disjoint union of the sets

$$\{w_1 b w_2 b w_3 \cdots b w_k : w_i \in \mathcal{L}^{n_i - 1}, n_i \geq 1, i \in [k], n_1 + n_2 + \cdots + n_k = n\},$$

for all $k \geq 1$, and each one has cardinality $[x^n]A(x)^k$. Applying this argument repeatedly, one can conclude that if a_n counts words in $\mathcal{L}^{n-1}$, then $\mathcal{I}^m(a_n)$ counts words in $(\mathcal{L} \cup \{b\})^{n-1}$, where the letter b is colored with m possible colors, whereas $[x^n](\mathcal{I}^{m-1}A(x))^k$ is the number of these words having exactly $k-1$ occurrences of the color m.

Generating functions

Based on these observations, Jancić [7] sets $a_n = \mathbb{F}_n$, where $(\mathbb{F}_n)_{n\geq 1}$ is the sequence of the Fine numbers (seq. A000957 in the OEIS), counting hill-free Dyck paths of semilength $n-1$, obtaining

$$\mathcal{I}^m(\mathbb{F}_n) = |\mathcal{D}_{n-1}^{(m)}|.$$

In terms of generating functions, if

$$F := F(x) := \sum_{n\geq 0} \mathbb{F}_{n+1} x^n$$

and

$$C := C(x) := \frac{1-\sqrt{1-4x}}{2x} = \sum_{n\geq 0} C_n x^n,$$

then

$$F(x) = \frac{xC(x)}{1+xC(x)}.$$

Generating functions

Applying the IT, for $m \in [3]$, we obtain

- $[x^n]\mathcal{I}F = [x^n]xC = C_{n-1}$,
  counting Dyck paths of semilength $n-1$,
- $[x^n]\mathcal{I}^2F = [x^n]xC^2 = C_n$,
  counting 2-Motzkin paths of length $n-1$,
- $[x^n]\mathcal{I}^3F = [x^n]\dfrac{xC^2}{1-xC^2} = \dbinom{2n-1}{n}$,
  counting 2-Motzkin prefixes of length $n-1$, or ballot paths of length $2n-1$, or pairs of noncrossing paths of length n.

Generating functions

In general, setting

$$F_m(x) := \mathcal{I}^m F = \mathcal{I}^m \frac{xC}{1+xC} = \frac{xC}{1-(m-1)xC}$$
$$= \sum_{k\geq 0}(m-1)^k x^{k+1} C^{k+1},$$

we can evaluate the coefficients of the m-th IT for all $m \in \mathbb{N}$:

$$[x^n]F_m(x) = [x^n]\sum_{k=0}^{n-1}(m-1)^k x^{k+1} C^{k+1} = \sum_{k=0}^{n-1}(m-1)^k [x^{n-k-1}] C^{k+1}$$
$$= \sum_{k=0}^{n-1}(m-1)^k \frac{k+1}{2n-k-1}\binom{2n-k-1}{n-k-1}$$
$$\overset{j=n-k-1}{=} \sum_{j=0}^{n-1}(m-1)^{n-j-1}\frac{n-j}{n+j}\binom{n+j}{n}.$$

Thank you!

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