Logical complexity of induced subgraph isomorphism for certain graph families

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September 10, 2020

For given graphs F, H let

- $v(F)$ be the number of its vertices,
- $\mathcal{I}(F)$ denote the class of all graphs containing a copy of F as an induced subgraph,
- $D[F]$ denote the minimum quantifier depth of a sentence in first-order logic with adjacency and equality relations ($\sim$ and $=$) that defines $\mathcal{I}(F)$
- $F \sqcup H$ denote the disjoint union of isomorphic copies of F and H, and for $m \in \mathbb{N}$ denote $mF = \bigsqcup_{i=1}^{m} F$

Goal

Given F or a graph sequence F_n, estimate $D[F]$ or asymptotic of $D[F_n]$ respectively.

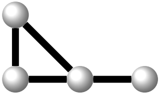
Motivation

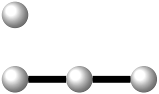
Time complexity of finding the value of fixed formula φ with depth d on an arbitrary graph G on n vertices is $O(n^d)$. If φ defines $\mathcal{I}(F)$, this gives us the same upper bound for time complexity of induced subgraph isomorphism — $ISI(F)$.

Nešetřil and Poljak showed that $ISI(F)$ is solvable in time $O(n^{\frac{w}{3}v(F)+\frac{2}{3}})$, where $w < 2.373$ is the exponent of fast square matrix multiplication. Given known bounds on $D[F]$, it is possible that the time bound of $O(n^{D[F]})$ can be better for some graphs F.

Known results

- $D[F] \le v(F)$, $D[F] = D[\hat{F}]$, where $\hat{F}$ is the complement of graph F.
- $D[K_n] = n$
- [O. Verbitsky, M. Zhukovskii, 2018] $D[F] \ge \max\left\{\lfloor \frac{1}{2} v(F) - 2\log_2 v(F) + 3 \rfloor, \chi(F), \frac{e(F)}{v(F)} + 2\right\}$, where $\chi(F)$ is the chromatic number of F, and $e(F)$ is the number of edges is F.
- [O. Verbitsky, M. Zhukovskii, 2018] Exact value of $D[F]$ is known for graphs F such that $v(F) \le 4$.
- The only known graphs such that $D[F] < v(F)$ is the paw graph and it's complement, $P_3 \sqcup K_1$:





Theorem (1)

For an arbitrary graph H, denote $F = P_3 \sqcup K_1 \sqcup H$. Then $D[F] \leq v(F) - 1$

Theorem (2)

For $m \in \mathbb{N}$, denote $F = P_3 \sqcup mK_1$. Then $D[F] = v(F) - 1$

Theorem (3)

Let $n_1, \ldots, n_k \in \mathbb{N}, n_1 \leq \ldots \leq n_k$. Denote $F = mK_{n_1,\ldots,n_k}$. Then $D[F] = v(F)$.

Theorem (4)

For any graph F such that $v(F) = 5$, $D[F] \geq 4$.

Theorem (1)

For an arbitrary graph H, denote $F = P_3 \sqcup K_1 \sqcup H$. Then $D[F] \leq v(F) - 1$

Proof.

Let G be paw graph complement. This is the formula for G:

$$\exists x_1([\exists x_3 \exists x_4 : x_1 \nsim x_3 \wedge x_1 \nsim x_4 \wedge x_3 \nsim x_4] \wedge$$
$$[\exists x_2(\exists x_4 : x_1 \sim x_2 \wedge x_1 \nsim x_4 \wedge x_2 \nsim x_4) \wedge$$
$$(\exists x_3 : x_1 \sim x_2 \wedge x_2 \sim x_3 \wedge x_1 \nsim x_3)])$$

We will modify it to get formula for F. □

Theorem (1)

For an arbitrary graph H, denote $F = P_3 \sqcup K_1 \sqcup H$. Then $D[F] \leq v(F) - 1$

Proof.

Let $m = v(H)$, $P_H(y_1, \ldots, y_m)$ be the formula representing that vertices $y_1, \ldots, y_m$ form graph H and $P(x, y_1, \ldots, y_m) = \bigwedge_{i=1}^{m} x \nsim y_i$. Then this is the formula for F.

$$\exists y_1, \ldots, \exists y_m \exists x_1 (P_H(y_1, \ldots, y_m) \wedge P(x_1, y_1, \ldots, y_m)$$
$$[\exists x_3 \exists x_4 : x_1 \nsim x_3 \wedge x_1 \nsim x_4 \wedge x_3 \nsim x_4$$
$$\wedge P(x_3, y_1, \ldots, y_m) \wedge P(x_4, y_1, \ldots, y_m)] \wedge$$
$$[\exists x_2 (P(x_2, y_1, \ldots, y_m) \wedge \exists x_4 : x_1 \sim x_2 \wedge x_1 \nsim x_4 \wedge x_2 \nsim x_4 \wedge$$
$$P(x_4, y_1, \ldots, y_m)) \wedge$$
$$(\exists x_3 : x_1 \sim x_2 \wedge x_2 \sim x_3 \wedge x_1 \nsim x_3 \wedge P(x_3, y_1, \ldots, y_m))])$$

Ehrenfeucht theorem

For any two graphs G, H, $r \in \mathbb{N}$ Duplicator wins the Ehrenfeucht–Fraïssé game on graphs G, H with r rounds iff any sentence ϕ of quantifier depth no more than r is simultaneously true or false on both graphs G, H.

Lower bound estimation

To show that $D[F] > k$ it is sufficient to show that for some graphs G, G', such that G contains induced copy of G and G' does not, Duplicator wins the game with k rounds on these graphs.

Theorem (2)

For $m \in \mathbb{N}$, denote $F = P_3 \sqcup mK_1$. Then $D[F] = v(F) - 1$

Proof.

Let $G_m = P_4 \sqcup mP_2$. We will use $G = G_m$, $G' = G_{m-1}$. It can be shown that Duplicator wins in $v(F) - 2 = m + 1$ rounds. Then $D[F] \geq v(F) - 1$. Using the previous theorem, we get $D[F] = v(F) - 1$. $\square$

Theorem (3)

Let $n_1, \ldots, n_k \in \mathbb{N}, n_1 \leq \ldots \leq n_k$. *Denote* $F = mK_{n_1,\ldots,n_k}$. *Then* $D[F] = v(F)$.

Proof.

Let $G^m_{\ell,k} = (V, E)$, where

$$\begin{aligned} V =& \{(i,j,h) | 1 \leq i \leq \ell, 1 \leq j \leq k, 1 \leq h \leq m\} \\ E =& \{((i_1, j_1, h), (i_2, j_2, h)) | \forall i_1, j_1, i_2, j_2, h : i_1 \neq i_2 \wedge j_1 \neq j_2\} \cup \\ & \{((i, j_1, h_1), (i, j_2, h_2)) | \forall i, j_1, j_2, h_1, h_2 : h_1 \neq h_2\} \end{aligned}$$

One can see that $G^m_{l,k}$ is a slight modification of $mK_{\underbrace{\ell, \ldots, \ell}_{k}}$, with h being the copy number, and j being the number of part in the copy.

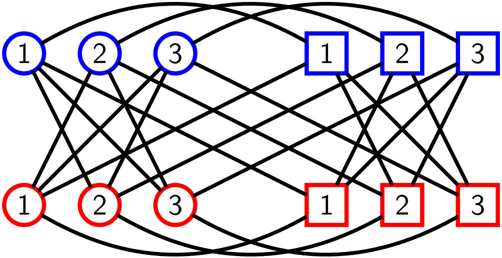


Figure: $G^2_{3,2}$, where vertices of the same shape correspond to the same copy (have the same h value), vertices of the same color correspond to the same part number (have the same j value), and the number of the vertex is its number i within the part.

Theorem (3)

Let $n_1, \ldots, n_k \in \mathbb{N}, n_1 \leq \ldots \leq n_k$. Denote $F = mK_{n_1,\ldots,n_k}$. Then $D[F] = v(F)$.

Proof.

Let $\ell = v(F)$, $G = G^m_{\ell,k}$, $G' = G^m_{\ell-1,k}$.
Then Duplicator wins in game on G, G' in $\ell - 1$ rounds. Since G contains induced copy of F, and G' does not, $D[F] \geq \ell = v(F)$. □

Theorem (4)

For any graph F such that $v(F) = 5$, $D[F] \geq 4$.

Proof.

Because $D[F] = D[\hat{F}]$, we have $D[F] \geq \frac{max\{e(F), \binom{v(F)}{2} - e(F)\}}{v(F)} + 2$, which gives us $D[F] \geq 4$ for any F with number of edges not equal to 5. Removing complements from consideration, we have only 4 graphs left. For each of them the bound is proven individually. $\square$