Turán problems for k-geodetic digraphs

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The classic Turán problem

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Theorem, Mantel, 1907

The largest size of a triangle-free graph with order n is $\lfloor \frac{n^2}{4} \rfloor$ and for order n the extremal graphs are complete bipartite graphs $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

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Turán generalised this result as follows:

Theorem, Turán,1941

A K_{r+1}-free graph with order n has at most $(1 - \frac{1}{r})\frac{n^2}{2}$ edges.

A Turán problem for cycles

K_3 is both a complete graph and a cycle. Erdős posed the following Turán-type problem.

Question

What is the maximum size of a graph with order n and no cycles of length $\leq r$?

Erdős conjectured that for $r = 4$ the answer is $(\frac{1}{2} + o(1))^{\frac{3}{2}} n^{\frac{3}{2}}$. If we denote the extremal size by $f(n)$, then it is only known that

$$\frac{1}{2\sqrt{2}} \leq \liminf_{n\to\infty} \frac{f(n)}{n^{\frac{3}{2}}} \leq \limsup_{n\to\infty} \frac{f(n)}{n^{\frac{3}{2}}} \leq \frac{1}{2}.$$

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What can we say about directed graphs?

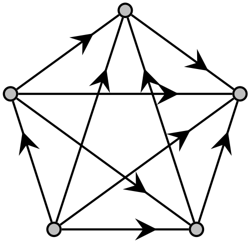
Excluding cycles in digraphs

Question

What is the largest size of a strongly connected digraph with girth $\geq g$?

We must include the restriction of strong connectivity as the acyclic tournament with order n has $\binom{n}{2}$ arcs but no cycles.

I can orient the edges in such a way that I get a digraph of order n, with $n(n-1)/2$ arcs, and **no directed cycles at all**.



Solution for cycles

This question was completely solved by Bermond et al (see 'Girth in digraphs').

Theorem

Let D be a strong digraph of order n, size m and girth g. Let $k \geq 2$. Then

$$m \geq \frac{1}{2}(n^2 + (3-2k)n + k^2 - k)$$

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This means that, asymptotically speaking, a strong digraph can have large girth and ‘almost all’ possible arcs present!

What is a k-geodetic digraph?

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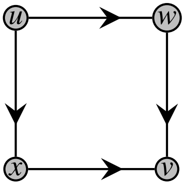
This digraph is not 2-geodetic.



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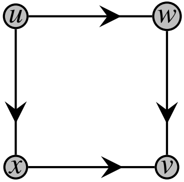
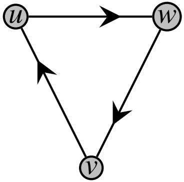


This digraph is not 3-geodetic.



Why are such digraphs interesting?

The **degree/geodecity problem** asks for the smallest possible order of a k-geodetic digraph with minimum out-degree d. It is known that the order n of such a digraph is bounded below by the directed Moore bound

$$n \geq M(d,k) = 1 + d + d^2 + \cdots + d^k.$$

The degree/geodecity problem is a generalisation of the undirected degree/girth problem. The first cages were identified by Tuite and Erskine.

The geodetic girth of a digraph G is the largest k such that G is k-geodetic. As an undirected graph has girth $\geq 2k+1$ if and only if it is k-geodetic (with suitable changes made to the definition) the geodetic girth of a digraph can be viewed as a 'girth-like' parameter.

An example of a cage

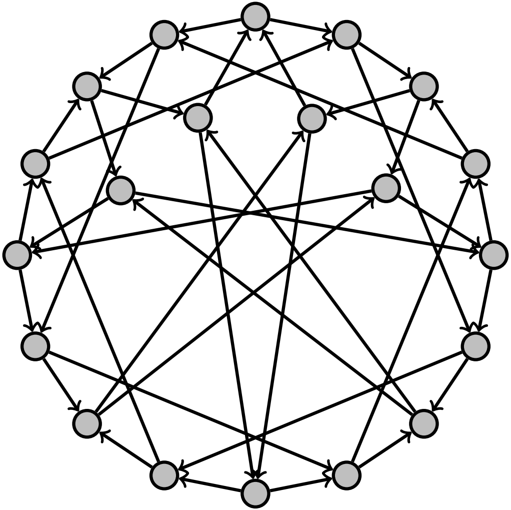


Figure: A smallest possible 3-geodetic digraph with out-degree 2

Problem statement

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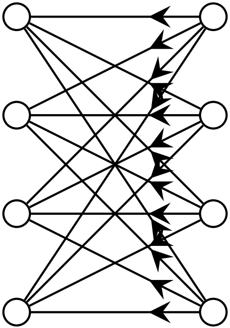
We can easily obtain a lower bound for $ex(n;k)$ by taking the complete bipartite graph $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ and orienting all arcs towards the same partite set.



Lower bound

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For $k \geq 2$ we have $ex(n;k) \geq \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$

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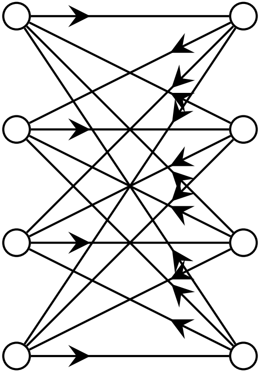
Note: For $k = 2$ and even n we can make this solution strongly connected by orienting a perfect matching in the opposite direction.



Counting Lemma

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For any $m \leq n-1$ we have $ex(n;k) \leq \frac{n(n-1)}{m(m-1)} ex(m;k)$.

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Proof

We count the pairs (F, e), where F is a subset of m vertices and e is an arc with both end-points in F. Let F be any subset of m vertices. In the induced subdigraph there can be at most $ex(m;k)$ arcs. Therefore there are at most $\binom{n}{m} ex(m;k)$ such pairs. For each arc e there are exactly $\binom{n-2}{m-2}$ subsets containing the endpoints of e, so it follows that

$$ex(n;2)\binom{n-2}{m-2} \leq \binom{n}{m} ex(m;k).$$

Rearranging yields the result.

Theorem

Theorem

For all $n \geq 4$, $n \geq k \geq 2$ we have $ex(n;k) = \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$.

Let $k = 2$. The theorem is easily shown to be true for $n = 4$. Let $n \geq 5$ and assume that the theorem is true for $n - 1$. Suppose that $n = 2r$ is even. Putting $m = n - 1$ in the counting lemma and using the induction hypothesis we have

$$ex(2r;2) \leq \frac{2r(2r-1)}{(2r-1)(2r-2)} r(r-1) = r^2$$

as required.

Now let $n = 2r + 1$. The counting lemma with $m = 2r$ gives

$$ex(2r+1;2) \leq \frac{2r(2r+1)}{2r(2r-1)} r^2 = \frac{(2r+1)r^2}{2r-1} < r^2 + r + 1,$$

so again the necessary inequality follows.

Classification of solutions

Let $n = 2r$. Let H be the underlying graph of an extremal digraph G.

If a vertex x has degree $< r$, then $G - x$ would have too many arcs. If any vertex has degree $> r$ then the size of G would be too large. Therefore H is r-regular.

Either H is bipartite or contains a triangle. Suppose that x, y, z form a triangle. H is diamond-free, so their neighbours outside the triangle are distinct. As H is r-regular it follows that $3(r-2) + 3 \leq 2r$, so that $r \leq 3$.

Hence $H \cong K_{r,r}$.
Likewise for $n = 2r + 1$ $H \cong K_{r,r+1}$.

In fact all solutions are obtained by orienting a matching in one direction and all other arcs in the other direction.

What if we require strong connectivity?

Definition

For $k \geq 2$ and $n \geq k$ let $ex^*(n;k)$ be the largest size of a strongly connected k-geodetic digraph with order n.

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We know from the orientations of complete bipartite graphs that $ex^*(2r;2) = r^2$.

However, it is easy to see that no strongly connected 2-geodetic digraphs with order $n = 2r+1$ and $ex(2r+1;2) = r^2 + r$ arcs exist!

A construction

It is easy to find a construction that yields $ex^*(2r+1;2) \geq r^2+2$.

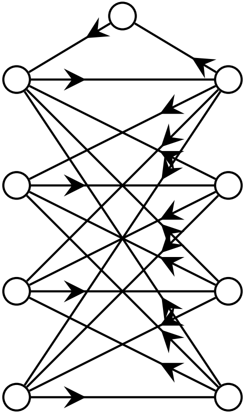

Figure: A strongly connected digraph with $n = 2r+1$ and $m = r^2+2$ (for $r = 4$)

In fact this is best possible!

Proof sketch

Theorem

$ex^*(2r;2) = r^2$ and $ex^*(2r+1;2) = r^2 + 2$.

Outline of proof for $n = 2r + 1$:

Let the size of an extremal digraph G be $m = r^2 + r - \epsilon$ for $0 \leq \epsilon \leq r - 3$.
Let H be the underlying graph of G. The maximum degree of H is $\Delta \geq r$.
For $\epsilon \leq r - 3$ G is bipartite.
A counting argument shows that:
$\epsilon \geq max\{|N^{+2}(x)|, |N^{-2}(x)|\}(min\{d^+(x), d^-(x)\} - 1)$
Use this to show that there are vertices x in G with out-degree $d^+(x) = r - 1$, in-degree $d^-(x) = 1$ and each out-neighbour has out-degree one. Derive a contradiction for $\epsilon \leq r - 3$.

Classification for $k = 2$

This analysis allows us to classify all strong 2-geodetic digraphs with order $n = 2r + 1$ and size $m = r^2 + 2$. Examples of these digraphs are shown on the following slides.

Theorem

If G is a 2-geodetic digraph with order $n = 2r + 1$, size $m = r^2 + 2$ and no sources or sinks, then G is either isomorphic to one of $A_r, B_{r,0}, B_{r,r-1}, C_r$ or D_r or is isomorphic to a member of the family $B_{r,t}, B'_{r,t}$ for some $1 \leq t \leq r - 2$. The digraphs in this list are mutually non-isomorphic and so there are $2r + 1$ distinct solutions up to isomorphism.

Strong digraphs with $n = 2r + 1$, $m = r^2 + 2$

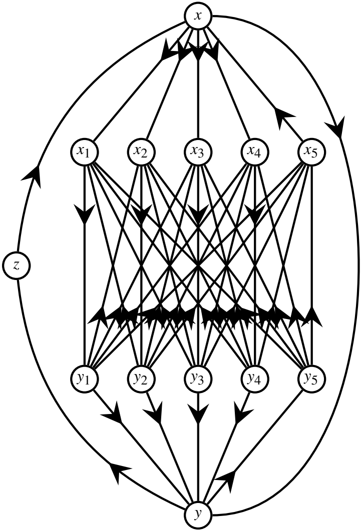

Figure: A_6

Strong digraphs with $n = 2r + 1$, $m = r^2 + 2$

This digraph is a member of a family of $t - 1$ solutions.

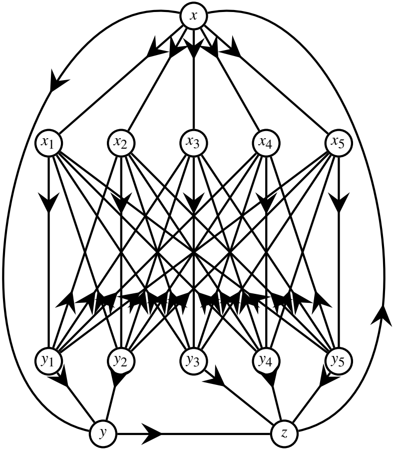

Figure: $B_{6,2}$

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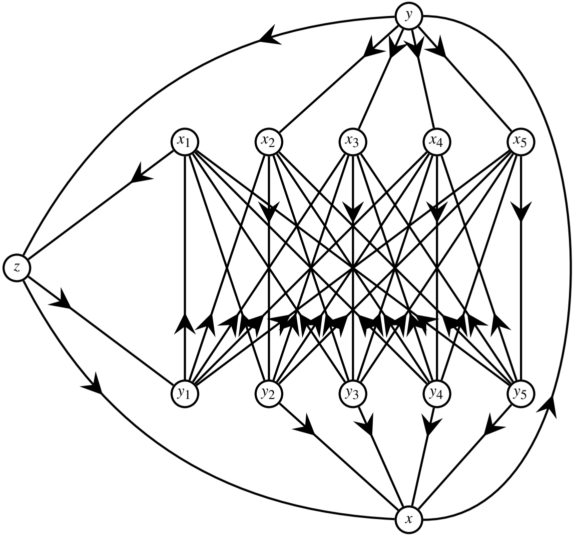

Figure: C_6

Strong digraphs with $n = 2r + 1$, $m = r^2 + 2$

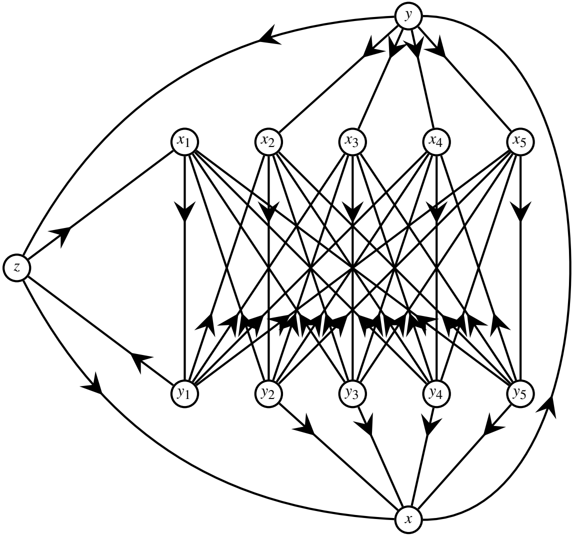

Figure: D_6

What about larger k?

Take the orientation of the complete bipartite graph $K_{r,r}$ with a perfect matching oriented in one direction and all other arcs in the opposite direction. Expand the arc in the perfect matching into paths of length $k-1$. This yields a strongly connected k-geodetic digraph $G_{k,r}$ with order kr, $r \geq 2$.

This gives the following lower bound:

Lemma

For $r \geq 2$ we have $ex^*(kr;k) \geq \frac{n^2}{k^2} + \frac{(k-2)n}{k}$.

Example

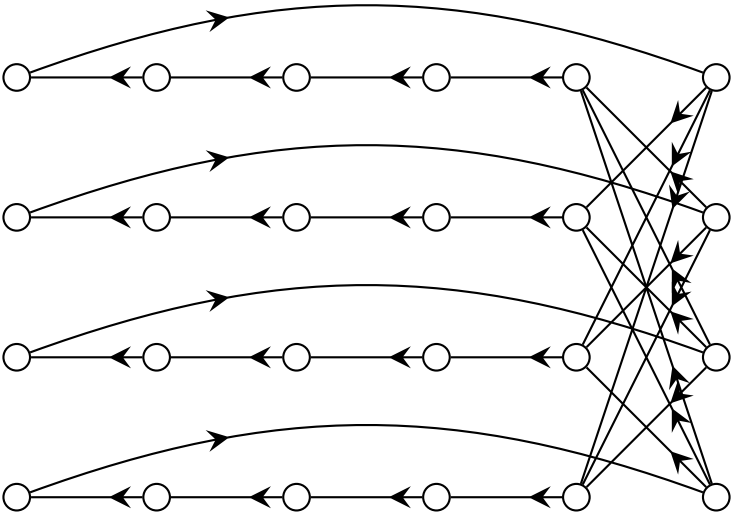

Figure: $G_{k,r}$ for $k = 6, r = 4$

More generally...

Let the quotient and remainder when n is divided by k be r and s respectively, i.e. $n = kr + s$. We assume that $s \leq r$.

Form the digraph $G(n, k)$ as follows. The vertex set of $G(n, k)$ consists of vertices $u_{i,j}$ for $1 \leq i \leq r$ and $1 \leq j \leq k$, as well as s further vertices $v_1, v_2, \ldots v_s$.

We define the adjacencies of $G(n, k)$ as follows.

i) $u_{i,j} \to u_{i,j+1}$ for $1 \leq i \leq r$ and $1 \leq j \leq k-1$

ii) $u_{i,k} \to v_i$ for $1 \leq i \leq s$

iii) $u_{i,k} \to u_{j,2}$ for $s+1 \leq i \leq r$ and $1 \leq j \leq s$

iv) $u_{i,k} \to u_{i',1}$ for $s+1 \leq i, i' \leq r$ and $i \neq i'$

More generally...

This digraph is k-geodetic and has size

$$m = rs + (k-1)r + s + (r-s)(r-1) = r^2 + (k-2)r + 2s.$$

If $r+1 \leq s \leq k-1$, then we have $\lfloor \frac{n}{k} \rfloor \leq k-2$, which is equivalent to $n \leq k^2 - k - 1$. Therefore these digraphs will certainly exist for $n \geq k^2 - k$. The arcs in part iii) can also be directed to $u_{j,2}$; combined with taking the converse of the resulting digraphs, this generates several isomorphism classes.

Let's see an example.

Example

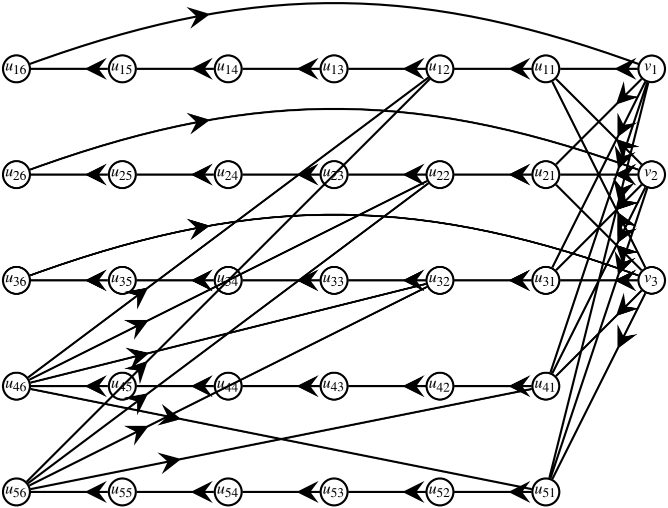

Figure: $G(33, 6)$

Conjecture

This construction generalises the digraphs $G_{k,r}$. Computer search confirms that these digraphs are extremal for any n and k in the range
$k = 3$ and $7 \leq n \leq 14$,
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for which the construction is defined. Moreover, for any n, k in this range such that $n = kr$ not only is the digraph $G_{k,r}$ extremal, it is the **unique solution** (subject to taking the converse etc)! This leads us to the following conjecture:

Conjecture

If $n \geq k+1$ and $n \leq (k+1)\lfloor \frac{n}{k} \rfloor$ (in particular for $n \geq k^2 - k$),

$$ex^*(n;k) = \lfloor \frac{n}{k} \rfloor^2 - (k+2)\lfloor \frac{n}{k} \rfloor + 2n.$$

Thank you!