Almost all optimally coloured complete graphs contain a rainbow Hamilton path

S. Gould T. Kelly D. Kühn D. Osthus

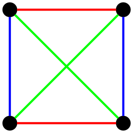
UNIVERSITY OF BIRMINGHAM

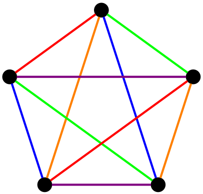
8th Polish Combinatorial Conference,
14th-18th September 2020

Chromatic index of the complete graph

Fact

If n is odd, then $\chi'(K_n) = n$.
If n is even, then $\chi'(K_n) = n - 1$, and the colour classes of an optimal edge-colouring form a '1-factorization' of K_n; that is, a decomposition into perfect matchings.

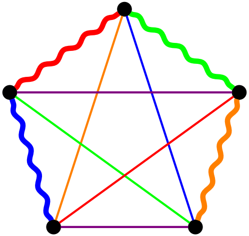




Rainbow Hamilton paths

Definition: Rainbow

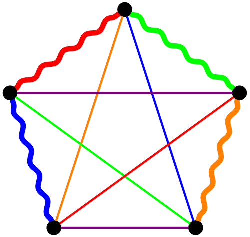
A subgraph H of an edge-coloured graph is called rainbow if all edges of H have distinct colours.



Rainbow Hamilton paths

Definition: Rainbow

A subgraph H of an edge-coloured graph is called rainbow if all edges of H have distinct colours.



Question

Whenever we properly colour the edges of K_n, is there always a rainbow Hamilton path?

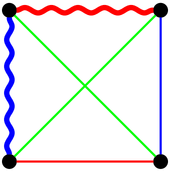
Rainbow Hamilton paths

Question

Whenever we properly colour the edges of K_n, is there always a rainbow Hamilton path?

Answer

No. Maamoun and Meyniel (1984) proved the existence of a 1-factorization of K_n (for $n \geq 4$ being any power of 2) with no rainbow Hamilton path.



Andersen's Conjecture

Andersen's Conjecture (1989)

Every properly edge-coloured K_n has a rainbow path of length $n-2$.

Andersen's Conjecture

Andersen's Conjecture (1989)

Every properly edge-coloured K_n has a rainbow path of length $n-2$.

What's known:

Trivial: There is a rainbow path of length $n/2-1$.

Gyárfás-Mhalla ('10): Every 1-factorization of K_n has a rainbow path of length $2n/3+1$.

Gyárfás-Ruszinkó-Sárközy-Schelp ('11): Every properly edge-coloured K_n has a rainbow path of length $(4/7-o(1))n$.

Gebauer-Mousset ('12) & Chen-Li ('15): ... $(3/4-o(1))n$.

Alon-Pokrovskiy-Sudakov ('17): ... $n-O(n^{3/4})$.

Balogh-Molla ('17): ... $n-O(\log n\sqrt{n})$.

Main results

Ferber-Jain-Sudakov ('20): Do 'almost all' 1-factorizations of K_n have a rainbow Hamilton path? **We prove:** Yes.

Main results

Ferber-Jain-Sudakov ('20): Do 'almost all' 1-factorizations of K_n have a rainbow Hamilton path? **We prove:** Yes.

Theorem (G., Kelly, Kühn, Osthus, 2020++)

As $n \to \infty$, the proportion of 1-factorizations of K_n that have a rainbow Hamilton path tends to one.

Main results

Ferber-Jain-Sudakov ('20): Do 'almost all' 1-factorizations of K_n have a rainbow Hamilton path? **We prove:** Yes.

Theorem (G., Kelly, Kühn, Osthus, 2020++)

As $n \to \infty$, the proportion of 1-factorizations of K_n that have a rainbow Hamilton path tends to one.

- I.e., Andersen's Conjecture holds in a strong sense for almost all 1-factorizations.
- Equivalently, almost all 1-factorizations of K_n have a Hamilton cycle using all colours – confirms strong version of a conjecture of Akbari, Etesami, Mahini, and Mahmoody for almost all 1-factorizations.

Main results

Ferber-Jain-Sudakov ('20): Do 'almost all' 1-factorizations of K_n have a rainbow Hamilton path? **We prove:** Yes.

Theorem (G., Kelly, Kühn, Osthus, 2020++)

As $n \to \infty$, the proportion of 1-factorizations of K_n that have a rainbow Hamilton path tends to one.

- I.e., Andersen's Conjecture holds in a strong sense for almost all 1-factorizations.
- Equivalently, almost all 1-factorizations of K_n have a Hamilton cycle using all colours – confirms strong version of a conjecture of Akbari, Etesami, Mahini, and Mahmoody for almost all 1-factorizations.

Theorems (G., Kelly, Kühn, Osthus, 2020++)

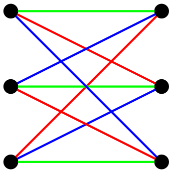
- Almost all 1-factorizations have a rainbow cycle using all the colours.
- For n odd, almost all optimal edge-colourings have a rainbow Hamilton cycle.

Latin squares and transversals

Latin square: An $n \times n$ array of n symbols such that each row and each column contains one instance of each symbol.

- Latin squares correspond to 1-factorizations of $K_{n,n}$. (Identify the vertex classes with rows and columns, and colours with symbols.)

1	2	3
3	1	2
2	3	1



Latin squares and transversals

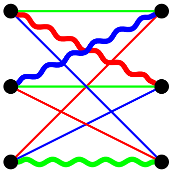
Latin square: An $n \times n$ array of n symbols such that each row and each column contains one instance of each symbol.

- Latin squares correspond to 1-factorizations of $K_{n,n}$. (Identify the vertex classes with rows and columns, and colours with symbols.)

Transversal: A collection of n cells, one from each row and column, containing one instance of each symbol.

- Transversals correspond to rainbow perfect matchings.

1	2	3
3	1	2
2	3	1



Latin squares and transversals

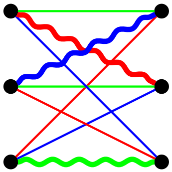
Latin square: An $n \times n$ array of n symbols such that each row and each column contains one instance of each symbol.

- Latin squares correspond to 1-factorizations of $K_{n,n}$. (Identify the vertex classes with rows and columns, and colours with symbols.)

Transversal: A collection of n cells, one from each row and column, containing one instance of each symbol.

- Transversals correspond to rainbow perfect matchings.

1	**2**	3
3	1	2
2	3	**1**



Ryser-Brualdi-Stein conj: Every LS has a partial transversal of size $n - 1$.

Kwan (2016+): Almost all Latin squares have a full transversal – 'partite analogue' of our result.

Symmetric Latin squares and Hamilton transversals

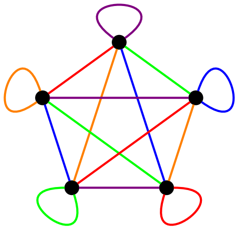
Symmetric Latin square: A Latin square whose entries are symmetric across the leading diagonal; that is, $a_{ij} = a_{ji}$ for all i, j, where a_{ij} denotes the symbol in cell $(i.j)$.

Symmetric Latin squares and Hamilton transversals

Symmetric Latin square: A Latin square whose entries are symmetric across the leading diagonal; that is, $a_{ij} = a_{ji}$ for all i, j, where a_{ij} denotes the symbol in cell $(i.j)$.

- Suppose n is odd. Symmetric $n \times n$ Latin squares correspond to optimal edge-colourings of $G = K_n$ + a loop at each vertex. (Put $V(G) = \{1, 2, \dots, n\}$ and colour edge ij with a colour corresponding to the symbol a_{ij}.)

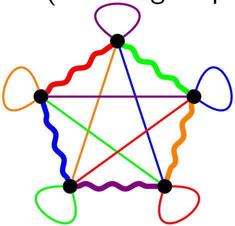
3	1	4	2	5
1	4	2	5	3
4	2	5	3	1
2	5	3	1	4
5	3	1	4	2



Symmetric Latin square: A Latin square whose entries are symmetric across the leading diagonal; that is, $a_{ij} = a_{ji}$ for all i, j, where a_{ij} denotes the symbol in cell $(i.j)$.

- Suppose n is odd. Symmetric $n \times n$ Latin squares correspond to optimal edge-colourings of $G = K_n +$ a loop at each vertex. (Put $V(G) = \{1, 2, \ldots, n\}$ and colour edge ij with a colour corresponding to the symbol a_{ij}.)
- Transversals correspond to rainbow 2-factors (including loops).

3	$\boxed{1}$	4	2	5
1	4	$\boxed{2}$	5	3
4	2	5	$\boxed{3}$	1
2	5	3	1	$\boxed{4}$
$\boxed{5}$	3	1	4	2

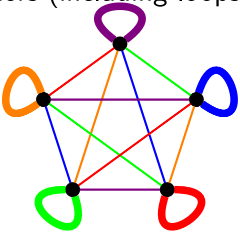


Symmetric Latin squares and Hamilton transversals

Symmetric Latin square: A Latin square whose entries are symmetric across the leading diagonal; that is, $a_{ij} = a_{ji}$ for all i, j, where a_{ij} denotes the symbol in cell $(i.j)$.

- Suppose n is odd. Symmetric $n \times n$ Latin squares correspond to optimal edge-colourings of $G = K_n +$ a loop at each vertex. (Put $V(G) = \{1, 2, \ldots, n\}$ and colour edge ij with a colour corresponding to the symbol a_{ij}.)
- Transversals correspond to rainbow 2-factors (including loops).

3	1	4	2	5
1	4	2	5	3
4	2	5	3	1
2	5	3	1	4
5	3	1	4	2

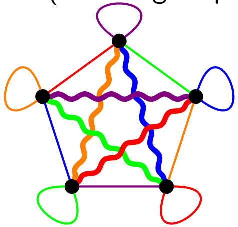


Symmetric Latin squares and Hamilton transversals

Symmetric Latin square: A Latin square whose entries are symmetric across the leading diagonal; that is, $a_{ij} = a_{ji}$ for all i, j, where a_{ij} denotes the symbol in cell $(i.j)$.

- Suppose n is odd. Symmetric $n \times n$ Latin squares correspond to optimal edge-colourings of $G = K_n$ + a loop at each vertex. (Put $V(G) = \{1, 2, \ldots, n\}$ and colour edge ij with a colour corresponding to the symbol a_{ij}.)
- Transversals correspond to rainbow 2-factors (including loops).

3	1	**4**	2	5
1	4	2	**5**	3
4	2	5	3	**1**
2	5	3	1	4
5	**3**	1	4	2

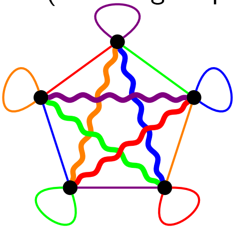


Symmetric Latin squares and Hamilton transversals

Symmetric Latin square: A Latin square whose entries are symmetric across the leading diagonal; that is, $a_{ij} = a_{ji}$ for all i, j, where a_{ij} denotes the symbol in cell $(i.j)$.

- Suppose n is odd. Symmetric $n \times n$ Latin squares correspond to optimal edge-colourings of $G = K_n$ + a loop at each vertex. (Put $V(G) = \{1, 2, \ldots, n\}$ and colour edge ij with a colour corresponding to the symbol a_{ij}.)
- Transversals correspond to rainbow 2-factors (including loops).

3	1	4	2	5
1	4	2	5	3
4	2	5	3	1
2	5	3	1	4
5	3	1	4	2



Corollary (GKKO): For n odd, almost all symmetric $n \times n$ Latin squares have a 'Hamilton transversal'. (The 2-factor is a Hamilton cycle.)

The big picture

Latin square	Proper edge-colouring of K_n

The big picture

Latin square	Proper edge-colouring of K_n
Ryser-Brualdi-Stein: Always an order $n-1$ "partial" transversal?	**Andersen:** Always a rainbow path of length $n-2$?

The big picture

Latin square	Proper edge-colouring of K_n
Ryser-Brualdi-Stein: Always an order $n-1$ "partial" transversal?	**Andersen:** Always a rainbow path of length $n-2$?
Best result in arbitrary Latin squares: Keevash-Pokrosvkiy-Sudakov-Yepremyan: $n - O(\log n / \log\log n)$	Best result in arbitrary proper edge-colourings of K_n: Balogh-Molla: $n - O(\log n\sqrt{n})$

The big picture

Latin square	Proper edge-colouring of K_n
Ryser-Brualdi-Stein: Always an order $n-1$ "partial" transversal?	**Andersen:** Always a rainbow path of length $n-2$?
Best result in arbitrary Latin squares: Keevash-Pokrosvkiy-Sudakov-Yepremyan: $n - O(\log n / \log \log n)$	Best result in arbitrary proper edge-colourings of K_n: Balogh-Molla: $n - O(\log n \sqrt{n})$
Kwan: Almost all Latin squares have transversal	**G-K-K-O:** Almost all 1-factorizations have rainbow Hamilton path

Proof strategy

Theorem (G., Kelly, Kühn, Osthus, 2020++)

A uniformly random 1-factorization of K_n has a rainbow Hamilton path with high probability.

The proof splits neatly into two halves. Suppose n is even.

Proof strategy

Theorem (G., Kelly, Kühn, Osthus, 2020++)

A uniformly random 1-factorization of K_n has a rainbow Hamilton path with high probability.

The proof splits neatly into two halves. Suppose n is even.

- Probabilistic analysis: Show that almost all 1-factorizations ϕ of K_n have some property, say Property $(\star)$.
- Construct the RHP: Show that any 1-factorization of K_n which has Property $(\star)$ admits a rainbow Hamilton path.

Proof strategy: Probabilistic Analysis

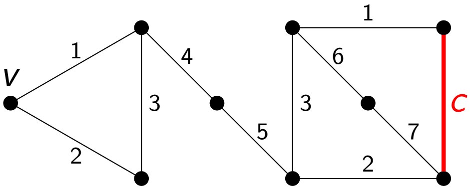
Property ($\star$)

'ϕ has Property ($\star$)' $\Leftrightarrow$ ϕ has many well-spread special 'absorbing' subgraphs.

Proof strategy: Probabilistic Analysis

Property $(\star)$

'ϕ has Property $(\star)$' $\Leftrightarrow$ ϕ has many well-spread special 'absorbing' subgraphs.

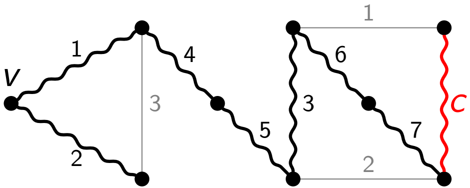


Each absorbing subgraph can 'absorb' a pair (v, c), where v is a vertex and c is a colour.

Proof strategy: Probabilistic Analysis

Property ($\star$)

'ϕ has Property ($\star$)' $\Leftrightarrow$ ϕ has many well-spread special 'absorbing' subgraphs.



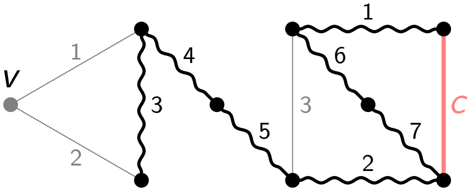
Each absorbing subgraph can 'absorb' a pair (v, c), where v is a vertex and c is a colour.

- The key property of absorbing subgraphs is that they have a path which uses all vertices and colours...

Proof strategy: Probabilistic Analysis

Property ($\star$)

'ϕ has Property ($\star$)' $\Leftrightarrow$ ϕ has many well-spread special 'absorbing' subgraphs.



Each absorbing subgraph can 'absorb' a pair (v, c), where v is a vertex and c is a colour.

- The key property of absorbing subgraphs is that they have a path which uses all vertices and colours...
- ... and a path using all vertices except v, and all colours except c.

Proof strategy: Probabilistic Analysis

Aim

Almost all 1-factorizations of K_n have Property $(\star)$.

Proof strategy: Probabilistic Analysis

Aim

Almost all 1-factorizations of K_n have Property $(\star)$.

Key challenge of the proof: Uniformly random choice of a 1-factorization ϕ of K_n is a 'rigid' probability space. There are many dependencies and correlations between events, and it is difficult to make local changes to ϕ without far-reaching, difficult-to-analyze effects.

Proof strategy: Probabilistic Analysis

Aim

Almost all 1-factorizations of K_n have Property $(\star)$.

Key challenge of the proof: Uniformly random choice of a 1-factorization ϕ of K_n is a 'rigid' probability space. There are many dependencies and correlations between events, and it is difficult to make local changes to ϕ without far-reaching, difficult-to-analyze effects.

- Ideally, one would directly analyze a uniformly random 1-factorization of K_n.
- Instead, we analyze a slightly different probability space that is less 'rigid'. (Delete all but a few colour classes.)
- ...and use known results to compare the two probability spaces.

Proof strategy: Constructing the RHP

Aim

1-factorization ϕ of K_n has Property $(\star)$ $\implies$ ϕ contains a rainbow Hamilton path.

Fix a 1-factorization ϕ of K_n which has Property $(\star)$.

Proof strategy: Constructing the RHP

Aim

1-factorization ϕ of K_n has Property $(\star)$ $\implies$ ϕ contains a rainbow Hamilton path.

Fix a 1-factorization ϕ of K_n which has Property $(\star)$.

1. Randomly partition the vertices, colours, and edges into a large 'main' slice, and a small 'absorbing' slice.

Proof strategy: Constructing the RHP

Aim

1-factorization ϕ of K_n has Property $(\star)$ $\implies$ ϕ contains a rainbow Hamilton path.

Fix a 1-factorization ϕ of K_n which has Property $(\star)$.

1. Randomly partition the vertices, colours, and edges into a large 'main' slice, and a small 'absorbing' slice.
2. Find a rainbow path using almost all of the main slice, by using results following from the 'Rödl Nibble'.

Proof strategy: Constructing the RHP

Aim

1-factorization ϕ of K_n has Property $(\star)$ $\implies$ ϕ contains a rainbow Hamilton path.

Fix a 1-factorization ϕ of K_n which has Property $(\star)$.

1. Randomly partition the vertices, colours, and edges into a large 'main' slice, and a small 'absorbing' slice.
2. Find a rainbow path using almost all of the main slice, by using results following from the 'Rödl Nibble'.
3. Finally, use Property $(\star)$ to show that the absorbing slice contains the absorbing subgraphs required to absorb the remaining vertices and colours into an RHP.

Paper Title

Almost all optimally coloured complete graphs contain a rainbow Hamilton path

Find the paper online at

https://arxiv.org/abs/2007.00395

Thank you for reading!