On a problem of Frankl and Füredi

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September 14-18, 2020
8^{th} Polish Combinatorial Conference

In extremal graph theory, we are interested in the following problem:

Problem 1

Let F be a family of graphs. What is the maximum number $ex(n, F)$ of edges in an F-free graph on n vertices?

One of the first results regarding this problem is Mantel's theorem (1907).

Theorem 1

The maximum number of edges in an n-vertex triangle-free graph is $\lfloor \frac{n^2}{4} \rfloor$. Furthermore, the only triangle-free graph with $\lfloor \frac{n^2}{4} \rfloor$ edges is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$

This problem is known as Tùran problem, for hypergraphs it can be stated as follows.

Problem 2

Let F be an r-uniform hypergraph. What is the maximum number $ex(n, F)$ of edges in an r-uniform hypergraph on n vertices that does not contain a copy of F?

The simplest case is when F consists of the complete r-uniform hypergraph on $r+1$ vertices. Nonetheless, it is extremely hard to solve this problem. In fact, no case was solved for $r > 2$.

Frankl and Füredi considered this particular problem

Problem 3

What is the maximum number of hyperedges in an r-uniform hypergraph with n vertices, such that every set of $r+1$ vertices contains 0 or exactly 2 hyperedges?

We call such hypergraphs, FF_r-hypergraphs. In [2], Frankl and Füredi solved the problem when $r = 3$.

Gunderson and Semeraro [3] considered Frankl and Füredi's problem for $r = 4$. They gave the following result.

Proposition 1

Let H be a 4-uniform hypergraph on n vertices with the property that every set of 5 vertices contains at most 2 hyperedges. Then, the maximum number of hyperedges is $\frac{n}{16}\binom{n}{3}$

In other words, they proved that

$$ex(n, K_5^{-2}) \leq \frac{n}{16}\binom{n}{3}$$

where K_5^{-2} is the unique 4-uniform hypergraph on 5 vertices with 3 hyperedges.

Example of FF_4 hypergraphs can be obtained from tournaments where the hyperedges correspond to the 4-subtournaments containing one 3-cycle. We call these 4-subtournaments diamonds. It follows that finding an upper bound on the number of diamonds yields a lower bound on the maximum number of hyperedges in FF_4 hypergraphs.
Gunderson and Semeraro showed that tournaments obtained by adding a vertex dominating a Paley tournaments have exactly $\frac{n}{16}\binom{n}{3}$ diamonds. Hence, their bound is exact when $n = q + 1$ for some prime power $q \equiv 0 \pmod 4$.

Bounds on the number of diamonds

In [1], we gave exact bounds on the number of diamonds δ_T. The first bound is the same as FF_4 hypergraphs.

Theorem 2

Let T be an n-tournament and S its skew adjacency matrix. Then

$$\delta_T \leq \frac{n}{16}\binom{n}{3}$$

Moreover, equality holds if and only if $S^2 = (1-n)I_n$.

Skew-symmetric ± 1-matrices verifying the last condition are called skew Conference matrices, they exist only if $n \equiv 0 \pmod 4$. Their existence is conjectured for every such order!

It follows from the last theorem that if there exists a skew Conference matrix of order n, then

$$ex(n, K^{-2}) = \frac{n}{16}\binom{n}{3}$$

This equality holds also for FF_4 hypergraphs.
We give improved bounds when n is not divisible by 4.

The case of $n \equiv 3 \pmod 4$

For $n \equiv 3 \pmod 4$, we have the following upper-bound.

Theorem 1

Let T be an n-tournament and S its Seidel adjacency matrix, such that $n \equiv 3 \mod 4$. Then

$$\delta_T \leq \frac{1}{96} n(n-1)(n-3)(n+1)$$

Moreover, equality holds if and only if $|S^2 + nI_n| = J_n$.

Proposition 2

$|S^2 + nI_n| = J_n$ if and only if T is switching equivalent to a doubly regular tournament.

For $n \equiv 3 \pmod 4$, we improve the bound on $ex(n, K_5^{-2})$.

Theorem 3

Let $n \equiv 3 \pmod 4$. *Then*

$$ex(n, K_5^{-2}) \leq \frac{1}{96} n(n-1)(n-3)(n+1)$$

Equality holds if there exists a skew-conference matrix of order $n+1$.

The case of $n \equiv 2 \mod 4$

Theorem 2

Let T be an n-tournament and S its skew adjacency matrix, such that $n \equiv 2 \mod 4$. Then

$$\delta_T \leq \frac{1}{96} n(n-3)(n-2)(n+2)$$

Moreover, equality holds if and only if

$$|S^2 - (n-3)I_n| = \begin{pmatrix} 2J_{n/2} & O \\ O & 2J_{n/2} \end{pmatrix}$$

Examples of tournaments verifying the last bound are EW-matrices and matrices obtained from a doubly regular tournament by removing any vertex. But they are not the only ones!

The case of $n \equiv 1 \mod 4$

As already mentioned, if T is an n-tournament such that its skew adjacency matrix is a skew conference matrix, then T has the maximum number of diamonds.
Moreover, tournaments obtained from T by removing one and two vertices have the maximum number of diamonds among tournaments with $n-1$ and $n-2$ vertices respectively.
We think that this fact holds when we remove three vertices.

The following proposition gives the expected upper bound for $n \equiv 1 \pmod 4$.

Proposition 3

Let T be a tournament on $n+1 \equiv 2 \mod 4$ vertices with the maximum number of diamonds. Then

$$\delta(T \setminus v) = \frac{1}{96}(n+3)(n-1)(n-2)(n-3)$$

for every $v \in V(T)$.

We also may obtain tournaments with the previous upper bound in a different way.

Theorem 4

Let T be an n-tournament obtained by adding a vertex in any way to a tournament whose skew adjacency matrix is a skew conference matrix. Then

$$\delta(T \setminus v) = \frac{1}{96}(n+3)(n-1)(n-2)(n-3)$$

Summary of the results

The following theorem summarizes the results we have seen.

Theorem 5

If there exists a skew Conference matrix of order n, then

$$ex(n, K_5^{-2}) = \frac{n}{16}\binom{n}{3}$$

$$ex(n-1, K_5^{-2}) = \frac{1}{96}n(n-1)(n-3)(n+1)$$

$$ex(n-2, K_5^{-2}) \geq \frac{1}{96}n(n-3)(n-2)(n+2)$$

$$ex(n-3, K_5^{-2}) \geq \frac{1}{96}n(n-3)(n-2)(n+2)$$

These bounds hold also for FF_4 hypergraphs.

Gunderson and Semeraro proved that a K_5^{-2}-free hypergraph has $\frac{n}{16}\binom{n}{3}$ hyperedges iff it is a $3-(n,4,n/4)$ design. That is, every set of 3 vertices occurs in exactly $n/4$ hyperedges.
If we remove 1, 2 and three vertices from a $3-(n,4,n/4)$ desing, we obtain hypergraphs with the same number of hyperedges as the bounds in Theorem 5. Based on this remark, we may state the following conjecture.

Conjecture 1

Let $\mathcal{H}$ be an FF_4-hypergraph with n vertices

1. *If $n \equiv 2 \pmod 4$ then $\mathcal{H}$ has at most $\frac{1}{96}n(n-3)(n+2)(n-2)$ hyperedges.*
2. *If $n \equiv 1 \pmod 4$ then $\mathcal{H}$ has at most $\frac{1}{96}(n-1)(n-2)(n-3)(n+3)$ hyperedges.*

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