About 3-uniform hypergraphs.

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8^{th} Polish Combinatorial Conference

8 PCC Septembre 14-18, 2020

Plan

1. Introduction
   - Definition of Hypergraph
   - Modules for hypergraphs
   - Modular partition for hypergraphs
2. Realization and decomposability
   - Realization and decomposability of hypergraphs
3. Realizability of 3-uniform hypergraphs

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3. Realizability of 3-uniform hypergraphs

Motivation introduction.

Definition of Hypergraphs

Definition

A hypergraph H is defined by a vertex set $V(H)$ and an edge set $E(H)$, where $E(H) \subseteq 2^{V(H)} \setminus \{\emptyset\}$. In the sequel, we consider only hypergraphs H such that

$$E(H) \subseteq 2^{V(H)} \setminus (\{\emptyset\} \cup \{\{v\} : v \in V(H)\}).$$

- The *complete hypergraph* H is given by $H = \left(V(H), 2^{V(H)}\right)$
- A hypergraph H is *empty*, if $E(H) = \emptyset$.

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Definition of Hypergraphs

k-Uniform hypergraph

Given $k \geq 2$, a hypergraph H is ***k*-*uniform*** if

$$E(H) \subseteq \binom{V(H)}{k}$$

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- If $k = 2$, an 2-uniform hypergraph is a **graph**.
- If $k = 3$, an 3-uniform hypergraph is a hypergraph whose edges are **triangles**.

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Subhypergraphs

Subhypergraph

We say that a hypergraph H' is a *sub-hypergraph* of H, if $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H)$.

Induced subhypergraph

Let H be a hypergraph. With each $W \subseteq V(H)$, we associate the subhypergraph $H[W]$ of H *induced by* W, which is defined on $V(H[W]) = W$ by

$$E(H[W]) = \{e \in E(H) : e \subseteq W\}$$

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Realization by tournaments

Tournaments

A tournament is an oriented complete graph. The $\mathbf{3}$-*cycle* is the tournament $\{\{0,1,2\},\{01,12,20\}\}$. Let T be a tournament, we denote by $C_3(T)$ the set of all $\mathbf{3}$-cycle of T.
Then $(V(T), C_3(T))$ is an $\mathbf{3}$-uniform hypergraph. It is said to be The *C_3-structure* of T.

Definition

Given a $\mathbf{3}$-uniforme hypergraph H, a tournament T, with $V(T) = V(H)$, *realizes* H if $H = C_3(T)$. We say also that T is a *realization* of H.

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Example of a realizable hypergraph

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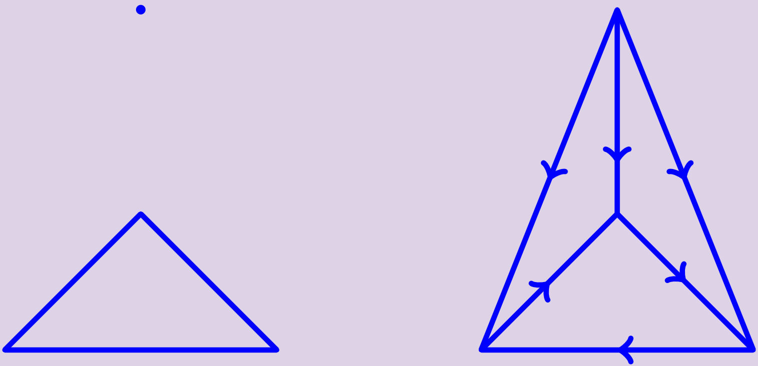
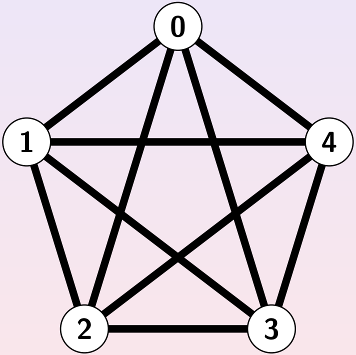


The distribution of 3-cycles in a tournament



Necessary condition

The **3**-uniform hypergraphs $(V(T), C_3(T))$ satisfy the following property:

$\mathcal{P}$: Each subset of $V(H)$ with **4** vertices contains $\mathbf{0}, \mathbf{1}$ or **2** hyper-edges.

Counter-example

Remark

There exist hypergraphs that satisfying the property $\mathcal{P}$ but they are not realisable.

Motivation introduction.

Notion of module for tournaments

Definition

Given a tournament $T = (V, A)$, a subset X of V is an interval of T provided that for every $a, b \in X$ and $x \in V \setminus X$, $(a, x) \in A$ if and only if $(b, x) \in A$.

For example, $\emptyset, \{\{x\} : x \in V\}$ and V are intervals of T, called *trivial intervals*. A tournament all the intervals of which are trivial is called *indecomposable*, otherwise, it is *decomposable*.

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Notion of module for tournaments

The 3-cycles of a tournament T are distributed with respect to intervals as follows.

Property

Given a tournament $T = (V, A)$, let M be an interval of T, and e be a 3-cycle of T, then we have one of the following three cases:

1. $e \cap M = \emptyset$ or ;
2. $e \subseteq M$ or ;
3. $e \cap M \neq \emptyset$.

Notion of module for tournaments

More precisely

1. for every $x, y \in M$ and $z \notin M$, then $\{x, y, z\}$ is not a $\mathbf{3}$-cycle;
2. for every $x, y \notin M$ and $z, z' \in M$, then $\{x, y, z\}$ is 3-cycle if and only if $\{x, y, z'\}$ is also 3-cycle.

Notion of module for hypergraphs

New definition

Let H be a hypergraph. A subset M of $V(H)$ is a module of H if for each $e \in E(H)$ such that $e \cap M \neq \emptyset$ and $e \setminus M \neq \emptyset$, there exists $m \in M$ such that $e \cap M = \{m\}$ and

$$\forall n \in M, (e \setminus \{m\}) \cup \{n\} \in E(H).$$

Classical definition

Let H be a hypergraph. A subset M of $V(H)$ is a module of H if for any $e, f \subseteq V(H)$ such that $|e| = |f|$, $e \setminus M = f \setminus M$, and $e \cap M \neq \emptyset$, we have

$$e \in E(H) \iff f \in E(H).$$

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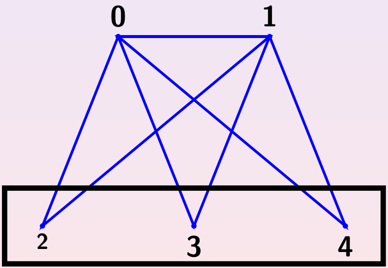
Motivation of choice between two defintions

Remark

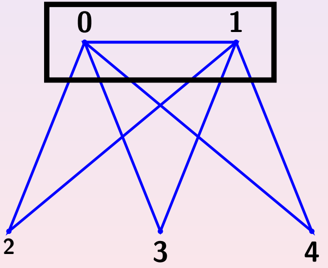
The classical defintion and our definition coincide for $\mathbf{2}$-uniform hypergraphs, that is, for graphs. They do not in the general case. Given a hypergraph $\boldsymbol{H}$, a module of $\boldsymbol{H}$ in the sense of Definition $\mathbf{1}$ is a module of the sense of Definition $\mathbf{2}$. the converse is not true. Given $\boldsymbol{n} \geq \mathbf{3}$, consider the 3-uniform hypergraph $\boldsymbol{H}$ defined by $\boldsymbol{V(H) = \{0, \ldots, n-1\}}$ and $\boldsymbol{E(H) = \{01p : 2 \leq p \leq n-1\}}$. In the sense of Definition $\mathbf{2}$, $\boldsymbol{\{0, 1\}}$ is a module of $\boldsymbol{H}$ wheras it is not a module of $\boldsymbol{H}$ in the sense of Definition $\mathbf{1}$

Our definition vs the classical definition

The set $\{2, 3, 4\}$ is a module in the sense of our definition



The set $\{0, 1\}$ is a module in the sense of the usual definition



Partitve family

We study the set of modules of hypergraphs.

Notation

Given a hypergraph H, the set of modules of H is denoted by $\mathcal{M}(H)$.
For instance, if H is an empty hypergrpah, then $\mathcal{M}(H) = 2^{V(H)}$.

Definition

Let S be set. A family $\mathcal{F}$ of subsets of S is a *partitve family* on S if it satisfies the following assertions.

1. $\emptyset \in \mathcal{F}$, $S \in \mathcal{F}$, and for every $x \in S$, $\{x\} \in \mathcal{F}$.
2. For any $M, N \in F$, $M \cap N \in \mathcal{F}$.
3. For any $M, N \in F$ such that $M \infty N$, then $M \cup N \in \mathcal{F}$ and $(M \setminus N) \cup (N \setminus M) \in \mathcal{F}$.

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Modular covering

Proposition

Given a hypergraph H, $\mathcal{M}(H)$ is a partitve family.

By the previous Proposition, $\emptyset$, $V(H)$ and $\{v\}$ where $v \in V(H)$, are modules of H, called *trivial*.

A hypergraph is *indecomposable* if all its modules are trivial, otherwise it is *decomposable*. An hypergraph H is *prime* if and only if it is *indecomposable* with $v(H) \geq 3$.

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Motivation introduction.

Theorem 1

The purpose of this section is to demonstrate the folllowing result.

Theorem 1

Given a hypergraph H with $v(H) \geq 2$, $H/\Pi(H)$ is an empty hypergraph, a prime hypergraph or a complete graph (i.e. $E(H/\Pi(H)) = \binom{\Pi(H)}{2}$.

Definition

Let H be a hypergraph. A partition P of $V(H)$ is a modular partition of H if $P \subseteq \mathcal{M}(H)$.

Definition

Given a modular partion P of H, the quotient H/P of H by P is defined on $V(H/P) = P$ as follows. For $\mathcal{E} \subseteq P$, $\mathcal{E} \in E(H/P)$ if $|\mathcal{E}| \geq 2$, and there exists $e \in E(H)$ such that $\mathcal{E} = \{X \in P : X \cap e \neq \emptyset\}$.

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Hypergraphs and decomposability

Remark

Consider a modular partition P of a hypergraph H. Let $e \in E(H)$ such that $|e/P| \geq 2$. Given $X \in e/P$, we have $e \cap X \neq \emptyset$, and $e \setminus X \neq \emptyset$ because $|e/P| \geq 2$. Since X is a module of H, we obtain $|e \cap X| = 1$. Therefore, e is a transverse of e/P. Moreover, since each element of e/P is a module of H, we obtain that each transverse of e/P is an edge of H. Given $\mathcal{E} \subseteq P$ such that $|\mathcal{E}| \geq 2$, it follows that $\mathcal{E} \in E(H/P)$ if and only if every transverse of $\mathcal{E}$ is an edge of H.

Remark

Lastly, consider a transverse t of P. The function θ_t from t to P, which maps each $x \in t$ to the unique element of P containing x, is an isomorphism from $H[t]$ onto H/P.

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Modular partition

We introduce the following strengthening of the notion of a module.

Strong module

Let H be a hypergraph. A module M of H is *strong* if for every module N of H, we have if $M \cap N \neq \emptyset$, then $M \subseteq N$ ou $N \subseteq M$.

Notation

We denote by $\Pi(H)$ the set of proper strong modules of H that are maximal under inclusion. Clearly, $\Pi(H)$ is a modular partition when $v(H) \geq 2$.

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Modular partition

In the next propositions, we study the links between the modules of a hypergraph with those of its quotients.

Proposition

Given a modular partition $\boldsymbol{P}$ of a hypergraph $\boldsymbol{H}$, the following two assertions hold

1. if $\boldsymbol{M}$ is a module of $\boldsymbol{H}$, then $\boldsymbol{M}/\boldsymbol{P}$ is a module of $\boldsymbol{H}/\boldsymbol{P}$;
2. if $\boldsymbol{\mathcal{M}}$ is a module of $\boldsymbol{H}/\boldsymbol{P}$, then $\cup\boldsymbol{\mathcal{M}}$ is a module of $\boldsymbol{H}$.

Modular partition

Proposition

Given a modular partition P of a hypergraph H, the following two assertions hold.

1. If M is a strong module of H, then M/P is a strong module of H/P.
2. Suppose that all the elements of P are strong modules of H. If $\mathcal{M}$ is a strong module of H/P, then $\cup\mathcal{M}$ is a strong module of H.

Connected Hypergraphs

Definition

A hypergraph H is connected if for distinct $v, \omega \in V(H)$, there exist a sequence $(e_0, \ldots, e_n)$ of edges of H, where $n \geq 0$, satisfying $v \in e_0$, $\omega \in e_n$, and (when $n \geq 1$) $e_i \cap e_{i+1} \neq \emptyset$ for every $0 \leq i \leq n-1$. Given a hypergraph H, a maximal connected subhypergraph of H is called a *component* of H.

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Given a hypergraph H, the set of the components of H is denoted by $\mathfrak{C}(H)$.

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Connected Hypergraphs

Remark

Let H be a hypergraph. For each component C of H, $V(C)$ is a module of H. Thus, $\{V(C) : C \in \mathfrak{C}(H)\}$ is a modular partition of H. Furthermore, for each component C of H, $V(C)$ is a strong module of H.

Lemma

Given a hypergraph H with $v(H) \geq 2$, the following assertions are equivalent :

1. H is disconnected;
2. H admits a modular bipartition P such that H/P is empty;
3. $\Pi(H) = \{V(C) : C \in \mathfrak{C}(H)\}, |\Pi(H)| \geq 2$, and $H/\Pi(H)$ is empty.

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Proof of Theorem 1

Let H be a hypergraph such that $v(H) \geq 2$. Because of the maximality of the elements of $\Pi(H)$, it follows from the second assertion of the previous Proposition that all the strong modules of $H/\Pi(H)$ are trivial. We establish the following result.

Theorem 2

Given a hypergraph H, all the strong modules of H are trivial if and only if H is an empty hypergraph, a prime hypergraph or a complete graph.

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Theorem 2

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Proof of Theorem 1

For a contradiction, suppose that $H/\Pi(H)$ admits a nontrivial strong module S. $\cup S$ is a strong module of H. Given $X \in S$, we obtain $X \subsetneq \cup S \subsetneq V(H)$, which contradicts the maximality of X. Consequently, all the strong modules of $H/\Pi(H)$ are trivial. To conclude, it suffices to apply **Theorem 2** to $H/\Pi(H)$.

Some useful notations

Let H be a hypergraph. As for tournaments, the set of the nonempty strong modules of H is denoted by $D(H)$. Clearly, $D(H)$ ordered by inclusion is a tree. It is called the *modular decomposition tree* of H. For convenience, set $D_{\geq 2}(H) = \{X \in D(H) : |X| \geq 2\}$. Moreover, we associate with each $X \in D_{\geq 2}(H)$, the label $\varepsilon_{H(X)}$ defined as follows:

$$\varepsilon_H(X) = \begin{cases} \triangle \text{ if } H[X]/\Pi(H[X]) \text{ is prime,} \\ o \text{ if } H[X]/\Pi(H[X]) \text{ is empty ,} \\ \bullet \text{ if } H[X]/\Pi(H[X]) \text{ is a complete graph.} \end{cases}$$

Modular partition of hypergraphs

Proposition

Given a hypergraph H, consider a strong module M of H. For every $N \subseteq M$, the following two assertions are equivalent:

1. N is a strong module of H;
2. N is a strong module of $H[M]$.

Motivation introduction.

Notation

Let H be a 3-uniform hypergraph. For $W \subseteq V(H)$ such that $W \neq \emptyset$, $\widetilde{W}^H$ denotes the intersection of the strong modules of H containing W. Note that $\widetilde{W}^H$ is the smallest strong module of H containing W.

Notation

Let T be a tournament. For a subset W of $V(T)$, set $\urcorner_T W = \{v \in V(T) \setminus W$ is not a module of $T[W \cup \{v\}]\}$ Consider a realizable and 3-uniform hypergraph. Let T be a realization of H. A module of T is clearly a module of H, but the converse is false. Nevertheless, we have the following result. Its proof is arduous and somewhat long, but it is central to establish Theorem 3.

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Key Proposition

The key proposition in this study is the following.

Key Proposition

Let H be a realizable and 3-uniform hypergraph. Consider a realization T of H. Let M be a module of H. If M is not a module of T, then the following four assertions hold.

1. $M \cup (\urcorner_T M)$ is a module of T;
2. M is not a strong module of H;
3. $M \cup (\urcorner_T M) \subseteq \widetilde{M}^H$;
4. $H(\widetilde{M}^H) = o$ and $\left|(H[\widetilde{M}^H])\right| \geq 3$.

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Let $\boldsymbol{H}$ be a realizable and $\mathbf{3}$-uniform hypergraph. Consider a realization $\boldsymbol{T}$ of $\boldsymbol{H}$. Let $\boldsymbol{M}$ be a module of $\boldsymbol{H}$. If $\boldsymbol{M}$ is not a module of $\boldsymbol{T}$, then the following four assertions hold.

1. $\boldsymbol{M} \cup (\urcorner_{\boldsymbol{T}} \boldsymbol{M})$ is a module of $\boldsymbol{T}$;
2. $\boldsymbol{M}$ is not a strong module of $\boldsymbol{H}$;
3. $\boldsymbol{M} \cup (\urcorner_{\boldsymbol{T}} \boldsymbol{M}) \subseteq \widetilde{\boldsymbol{M}}^{\boldsymbol{H}}$;
4. $\boldsymbol{H}(\widetilde{\boldsymbol{M}}^{\boldsymbol{H}}) = \boldsymbol{o}$ and $\left|(\boldsymbol{H}[\widetilde{\boldsymbol{M}}^{\boldsymbol{H}}])\right| \geq \mathbf{3}$.

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Key result

The next theorem is a key result. A realizable $\mathbf{3}$-uniform hypergraph and its realizations do not have the same modules, they share the same strong module.

Theorem 3

Consider a realizable and $\mathbf{3}$-uniform hypergraph $\boldsymbol{H}$. Given a realization $\boldsymbol{T}$ of $\boldsymbol{H}$, $\boldsymbol{H}$ and $\boldsymbol{T}$ share the same strong modules.

We establish Theorem **4** by using Theorems **2** and **3**.

Theorem 4

Consider a realizable **3**-uniform hypergraph H. For a realization T of H, we have H is prime if and only if T is prime.

Proof of Theorem 4

Suppose that H is prime. Since all the modules of T are modules of H, T is prime. Conversely, suppose that T is prime. Hence, all the strong modules of T are trivial. By Theorem **3**, all the strong modules of H are trivial. We obtain $\Pi(H) = \{\{v\} : v \in V(H)\}$ Thus, H is isomorphic to $H/\Pi(H)$. It follows from Theorem **1** that H is an empty hypergraph, a prime hypergraph or a complete graph. Since T is prime, we have $E(C_3(T)) \neq \emptyset$. Since $E(C_3(T)) = E(H)$, there exists $e \in E(H)$ such that $|e| = 3$. Therefore, H is not an empty hypergraph, and H is not a graph.It follows that H is prime.

We establish Theorem **4** by using Theorems **2** and **3**.

Theorem 4

Consider a realizable 3-uniform hypergraph H. For a realization T of H, we have H is prime if and only if T is prime.

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Characterization of realizable $\mathbf{3}$-uniform hypergraphs

Now, we characterize the realizable $\mathbf{3}$-uniform hypergraphs. To begin, we establish the following theorem by using the modular decomposition tree. We prove that a $\mathbf{3}$-uniform hypergraph is realizable if and only if all its prime, $\mathbf{3}$-uniform and induced subhypergraphs are realizable.

Theorem 5 (General case)

Given a $\mathbf{3}$-uniform hypergraph H, H is realizable if and only if for every $W \subseteq V(H)$ such that $H[W]$ is prime, $H[W]$ is realizable.

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Critical case

Theorem 6 (Critical case)

Given a critical and 3-uniform hypergraph H, H is realizable if and only if $v(H)$ is odd, and H is isomorphic to $C_3(T_{v(H)})$, $C_3(U_{v(H)})$ or $C_3(W_{v(H)})$.

Realizability of 3-uniform hypergraphs

The next proposition is useful to construct realizations from the modular decomposition tree. We need the following notation and remark.

Notation

Let H be a $3-$uniform hypergraph. We denote by $\mathcal{R}(H)$ the set of the realizations of H.

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Realizability of 3-uniform hypergraphs

Let H be a 3-uniform hypergraph. Consider $T \in \mathcal{R}(H)$. It follows from **Theorem 11** that $\mathcal{D}(H) = \mathcal{D}(T)$. For each $X \in \mathcal{D}_{\geq 2}(H)$, we have $\Pi(H[X]) = \Pi(T[X])$.
Therefore,

$$\forall X \in \mathcal{D}_{\geq 2}(H), T[X]/\Pi(T[X]) \in \mathcal{R}(H[X]/\Pi(H[X])).$$

Set,

$$\mathcal{R}_{\mathcal{D}}(H) = \bigcup_{X \in \mathcal{D}_{\geq 2}(H)} \mathcal{R}(H[X]/\Pi(H[X])).$$

Realizability of 3-uniform hypergraphs

We denote by $\delta_H(T)$ the function

$$\begin{array}{ccc} \mathcal{D}_{\geq 2}(H) & \longrightarrow & \mathcal{R}_{\mathcal{D}}(H) \\ Y & \longmapsto & T[Y]/\Pi(T[Y]) \end{array}$$

Lastly, denote by $\varphi(H)$ the set of the functions f from $\mathcal{D}_{\geq 2}(H)$ to $\mathcal{R}_{\mathcal{D}}(H)$ satisfying $\forall Y \in \mathcal{D}_{\geq 2}(H), f(Y) \in R(H[Y]/\Pi(H[Y]))$.
Under this notation, we obtain the function

$$\begin{array}{cccc} \delta_H : & R(H) & \longrightarrow & \varphi(H) \\ & T & \longmapsto & \delta_H(T) \end{array}$$

Realizability of 3-uniform hypergraphs

Proposition 2

For a **3**-uniform hypergraph $\boldsymbol{H}$, $\boldsymbol{\delta_H}$ is a bijection.

Theorem (Critical case)

Given a critical and **3**-uniform hypergraph $\boldsymbol{H}$, $\boldsymbol{H}$ is realizable if and only if $\boldsymbol{v(H)}$ is odd, and $\boldsymbol{H}$ is isomorphic to $\boldsymbol{C_3(T_{v(H)})}$, $\boldsymbol{C_3(U_{v(H)})}$ ou $\boldsymbol{C_3(W_{v(H)})}$.

Realizability of 3-uniform hypergraphs

Now, we characterize the non critical, prime and **3**-uniform hypergraphs that are realizable. We need the following notation.

Notation

Let H be a 3-uniform hypergraph. Consider a vertex ω of H. Set $V_\omega = V(H) \setminus \{\omega\}$. We denote by G_ω the graph defined on V_ω as follows. Given distinct elements v and v' of V_ω, $vv' \in E(G_\omega)$ if $\omega vv' \in E(H)$. Also, we denote by I_ω the set of the isolated vertices of G_ω.

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Realizability of 3-uniform hypergraphs

Notation

Let T be a tournament. Consider $W, W' \subseteq V(T)$ such that $W \cap W' = \emptyset$. We denote by $(W \twoheadrightarrow W')_T$ the subset of $\omega' \in W'$ such that there exists a sequence $\omega_0, \ldots, \omega_n$ satisfying

- $\omega_0 \in W$, $\omega_n = \omega'$;
- $\omega_1, \ldots, \omega_n \in W'$;
- for $i = 0, \ldots, n-1$, $\omega_i \omega_{i+1} \in A(T)$.

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Theorem (Prime, non critical case)

Let $\boldsymbol{H}$ be a non critical, prime, and **3**-uniform hypergraph. Consider a vertex ω of $\boldsymbol{H}$ such that $\boldsymbol{H}-\omega$ is prime. The **3**-uniform hypergraph $\boldsymbol{H}$ is realizable if and only if $\boldsymbol{H}-\omega$ admits a realization, say $\boldsymbol{T}_\omega$, satisfying the following two assertions:

(M1) There exists a bipartition $\{X, Y\}$ of $V_\omega \setminus I_\omega$ satisfying

(i) for each component C of G_ω, with $v(C) \geq 2$, C is bipartite with bipartition $\{X \cap V(C); Y \cap V(C)\}$;

(ii) $\forall x \in X, y \in Y$, we have $xy \in E(G_\omega) \iff xy \in A(T_\omega)$.

(M2) $(X \twoheadrightarrow I_\omega)_{T_\omega} \cap (Y \twoheadrightarrow I_\omega)_{(T_\omega)^*} = \emptyset$, $(X \twoheadrightarrow I_\omega)_{T_\omega} \cup (Y \twoheadrightarrow I_\omega)_{(T_\omega)^*} = I_\omega$. Furthermore, for $x \in X$, $y \in Y$, $x^+ \in (X \twoheadrightarrow I_\omega)_{T_\omega}$ and $y^- \in (Y \twoheadrightarrow I_\omega)_{(T_\omega)^*}$, we have $y^-x, yx^+; y^-x^+ \in A(T_\omega)$.

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Let $\boldsymbol{H}$ be a non critical, prime, and **3**-uniform hypergraph. Consider a vertex ω of $\boldsymbol{H}$ such that $\boldsymbol{H} - \omega$ is prime. The **3**-uniform hypergraph $\boldsymbol{H}$ is realizable if and only if $\boldsymbol{H} - \omega$ admits a realization, say $\boldsymbol{T_\omega}$, satisfying the following two assertions:

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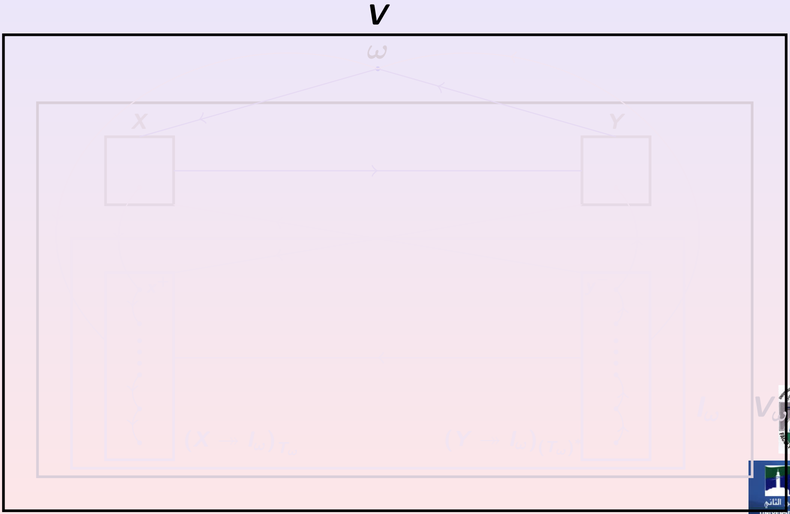
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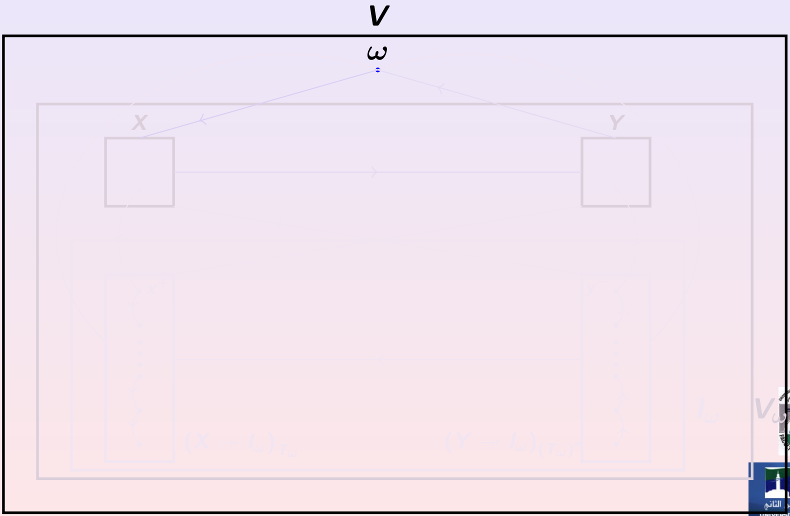
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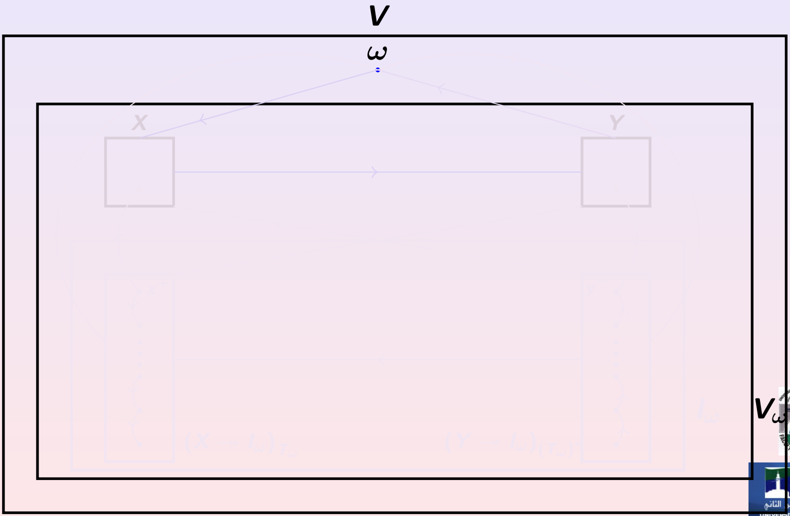
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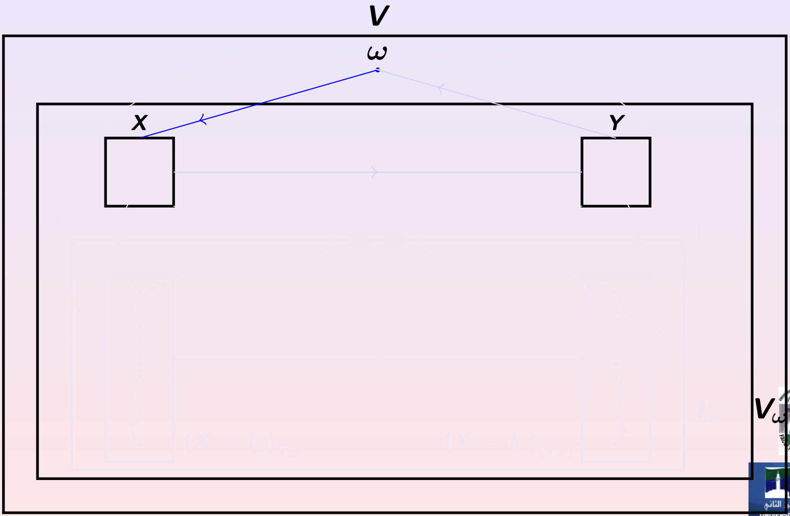
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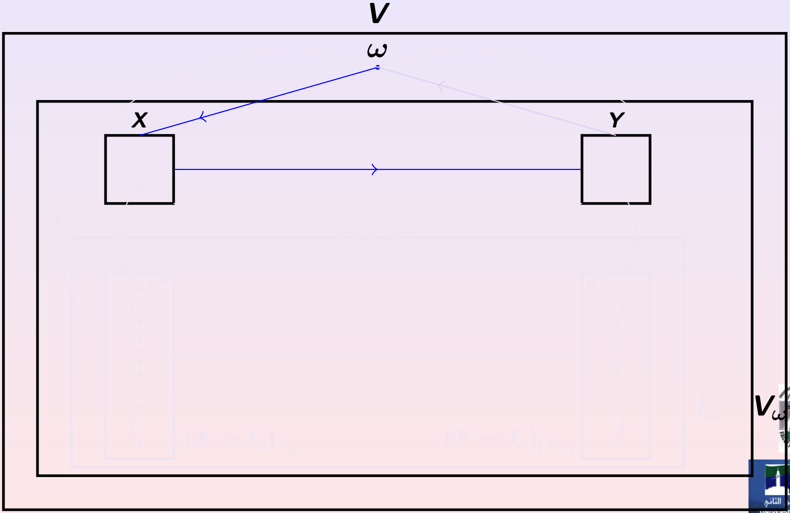
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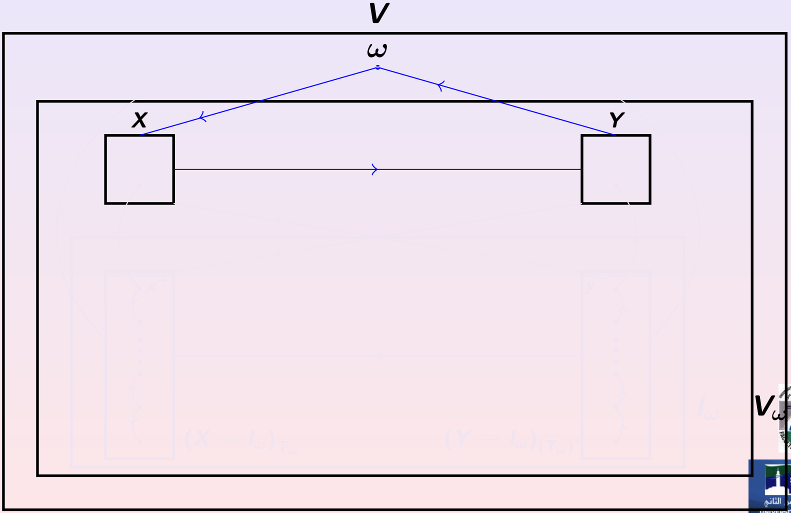
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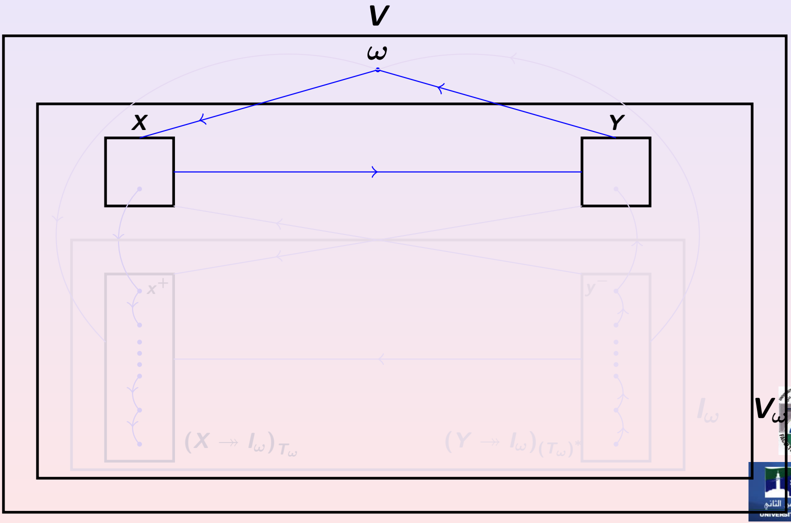
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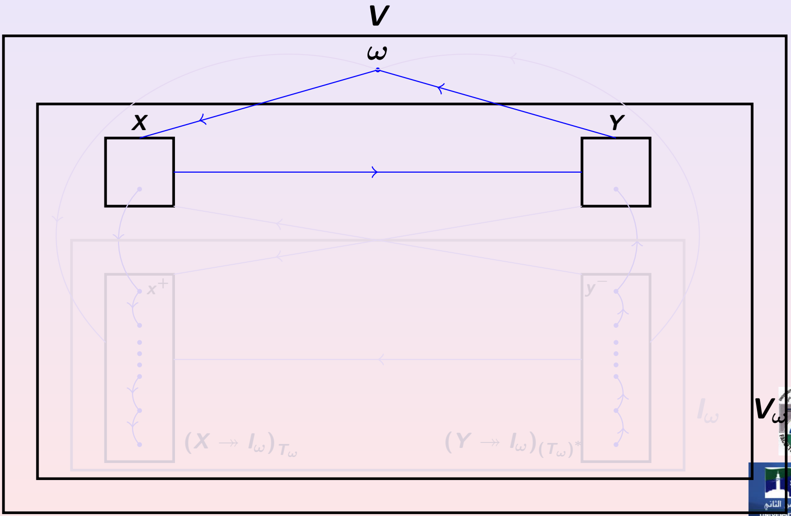
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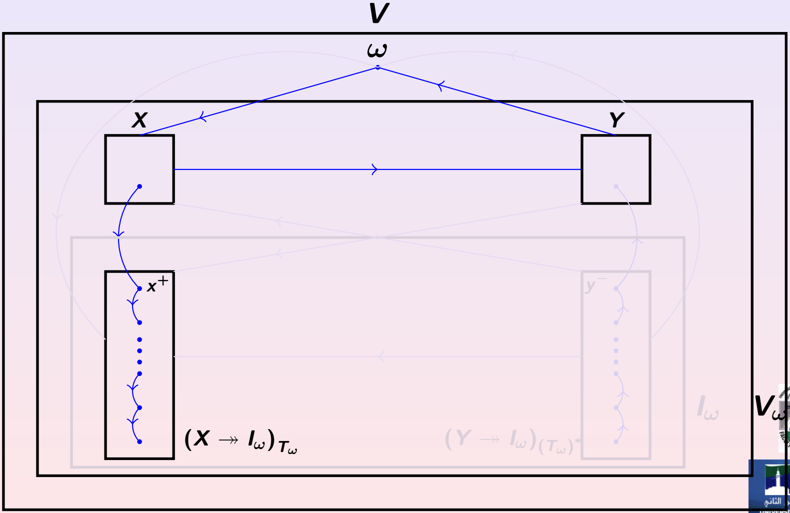
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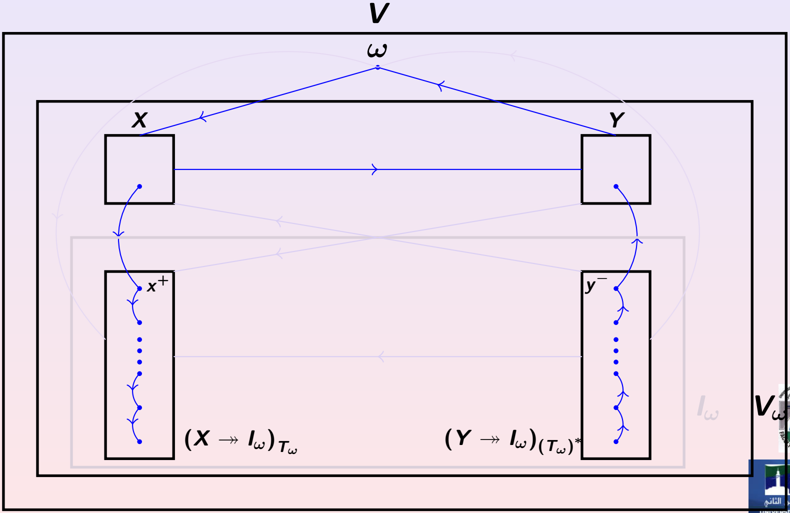
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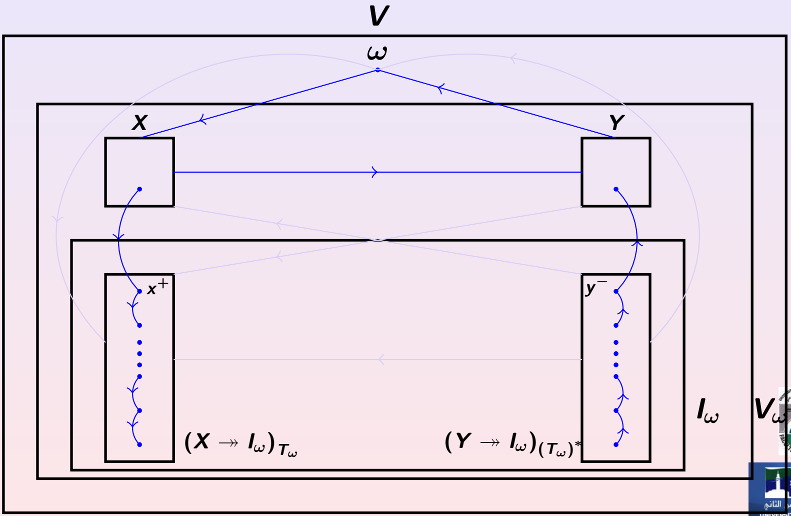
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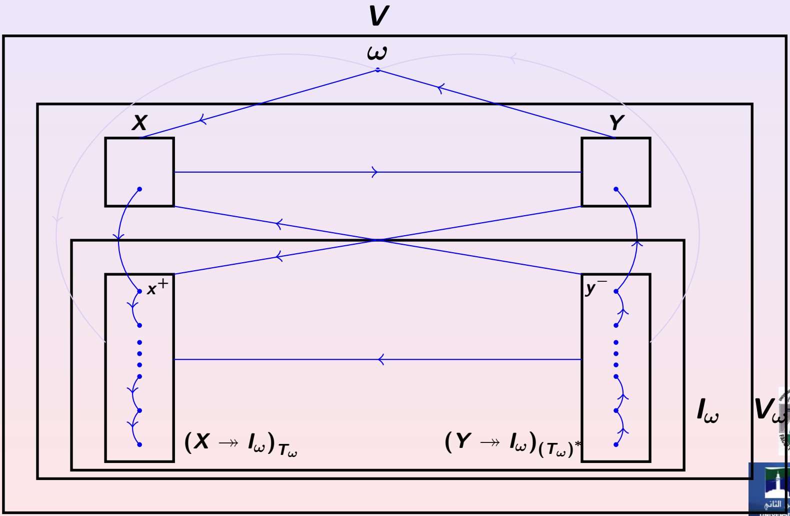
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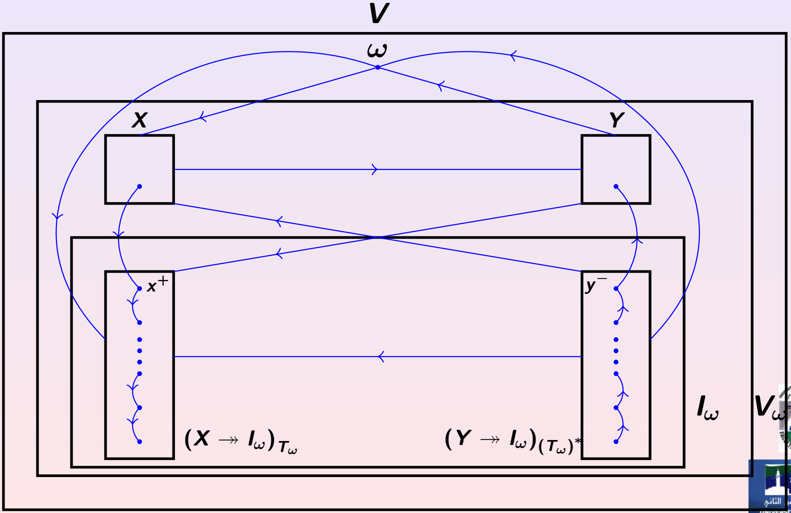
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Realizability of 3-uniform hypergraphs

We conclude by counting the number of realizations of a realizable and $\mathbf{3}$-uniform hypergraph. We need the following notation.

Notation

Let H be a 3-uniform hypergraph. Set

$$\begin{cases} \mathcal{D}_{\triangle}(H) = \{X \in \mathcal{D}_{\geq 2}(H) : \varepsilon_H(X) = \triangle\} \\ \text{and} \\ \mathcal{D}_{o}(H) = \{X \in \mathcal{D}_{\geq 2}(H) : \varepsilon_H(X) = o\} \end{cases}$$

Corollary

For a realizable and 3-uniform hypergraph, we have

$$|\mathcal{R}(H)| = 2^{|\mathcal{D}_{\triangle}(H)|} \times \prod_{X \in \mathcal{D}_{o}(H)} |\Pi(H[X]|!$$

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Thanks

Thank you!

Merci pour votre attention