Weak saturation stability for bipartite graphs

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Percolation in graphs and weak saturation

H — a spanning subgraph of G

- F-bootstrap percolation percolation:

  $H = H_0 \subset H_1 \subset \ldots \subset H_m = G$ such that $H_t \setminus H_{t-1}$ contains exactly one edge e_t and e_t creates a new copy of F

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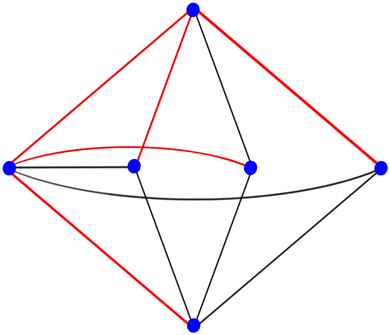
- H is weakly F-saturated in G if

  it does not contain any copy of F and it F-percolates in G

  w-sat(G, F) is the smallest number of edges in a weakly F-saturated graph in G

K_3-bootstrap percolation process



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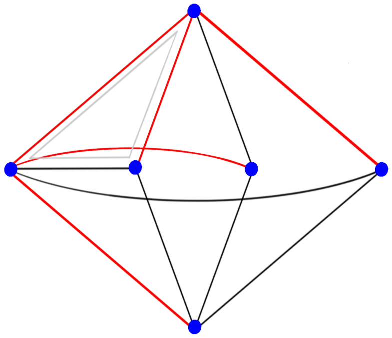

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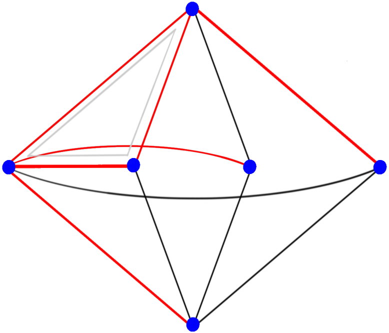

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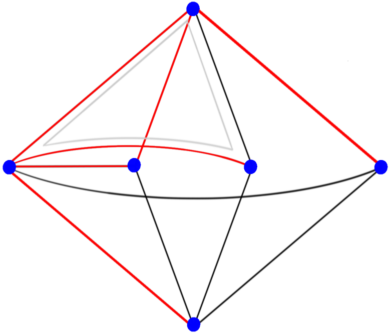

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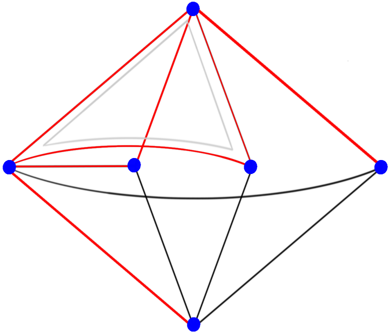

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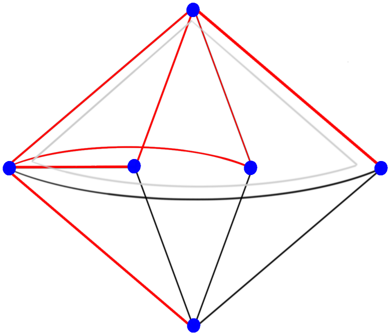

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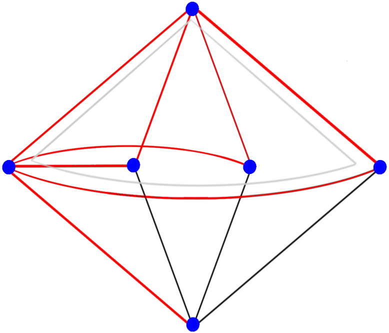

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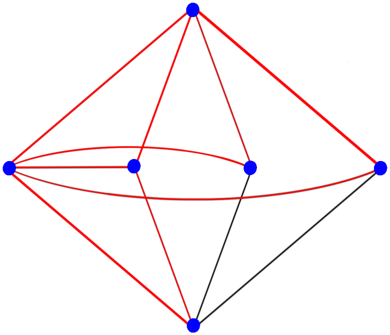

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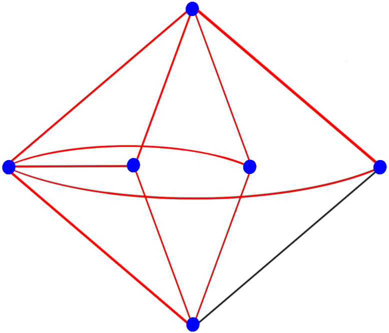

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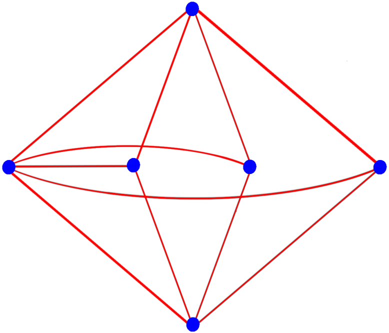

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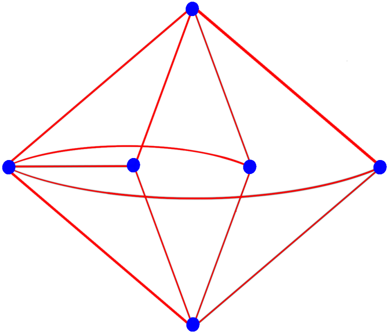

Weak saturation in complete graphs: triangle patterns

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What about w-sat(K_n, K_s)?

Weak saturation in complete graphs: clique patterns

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Weak saturation in complete graphs: clique patterns

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- Proved by Frankl in 1982, Kalai in 1984 and Lovász in 1977.

Stability of w-sat

Theorem (Korándi, Sudakov, 2017)

Let $0 < p < 1$ *be a constant,* $s \geq 3$*. Then*
$\text{w-sat}(G(n,p), K_s) = \text{w-sat}(K_n, K_s)$ *with high probability.*

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Problem: determine the exact probability range where the weak saturation number is $(s-2)n - \binom{s-1}{2}$.

Weak saturation in sparse random graphs

Theorem (Bidgoli, Mohammadian, Tayfeh-Rezaie, Z, 2020+)

There exists p_0 s.t.

- *if* $p \gg p_0$, *then* $\text{w-sat}(G(n,p), K_s) = \text{w-sat}(K_n, K_s)$ *w.h.p*
- *if* $p \ll p_0$, *then* $\text{w-sat}(G(n,p), K_s) \neq \text{w-sat}(K_n, K_s)$ *w.h.p.*

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If $p \geq n^{-\frac{1}{2s-3}} \ln^2 n$,

then $\text{w-sat}(G(n,p), K_s) = \text{w-sat}(K_n, K_s)$ *w.h.p.*

If $p \leq cn^{-\frac{2}{s+1}} (\ln n)^{\frac{2}{(s-2)(s+1)}}$,

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https://arxiv.org/abs/2006.06855

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     - A connected spanning subgraph of K_n is weakly K_3-saturated in K_n.
     - If a tree H is weakly K_3-saturated in $G(n,p)$, then it might contain an edge e that does not belong to a triangle.

   Then $H \setminus e$ is weakly K_3-saturated in $G(n,p) \setminus e$ which, in turn, is weakly K_3-saturated in $G(n,p)$. A contradiction since H has less then $n-1$ edges.

The proof of the upper bound for K_3

Let $p > 2\sqrt[3]{\frac{\ln n}{n}}$.

- w.h.p. any three vertices of $G(n, p)$ have a common neighbor,
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If G has the above properties, then it has a weakly K_3-saturated subtree.

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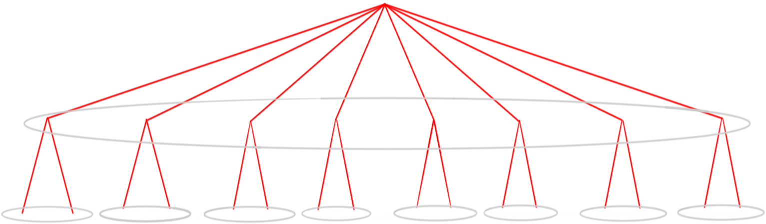

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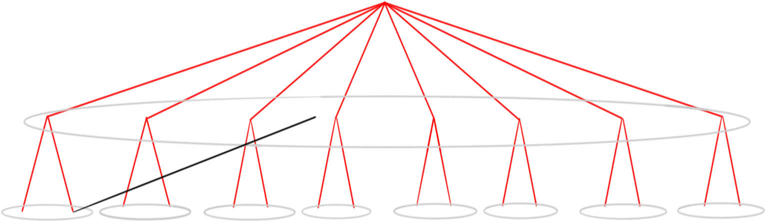

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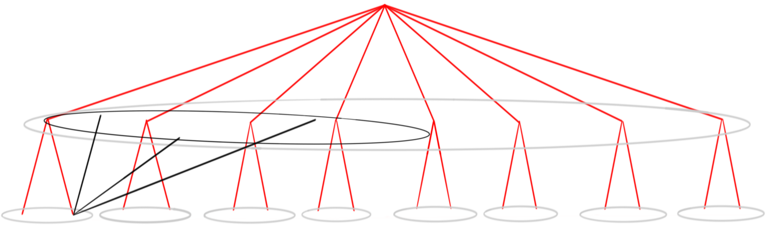

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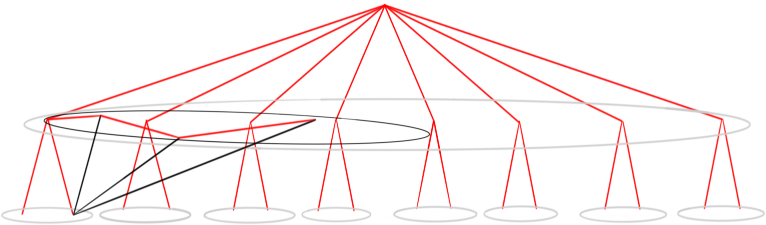

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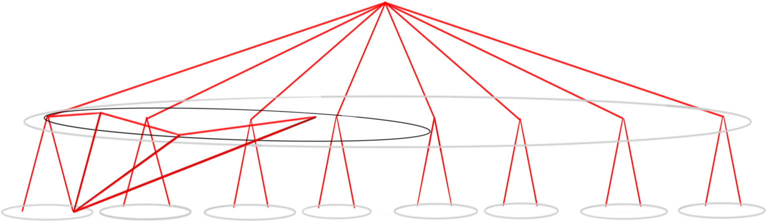

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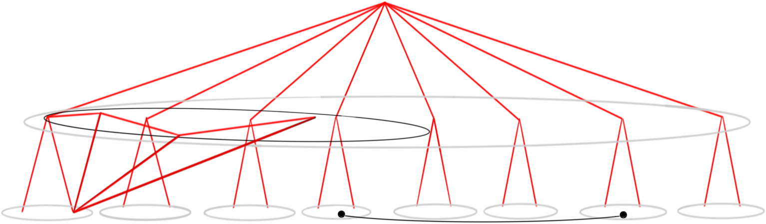

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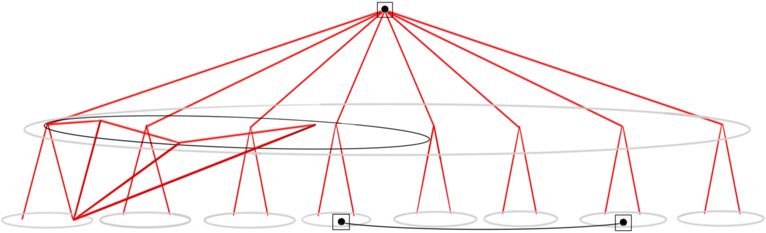

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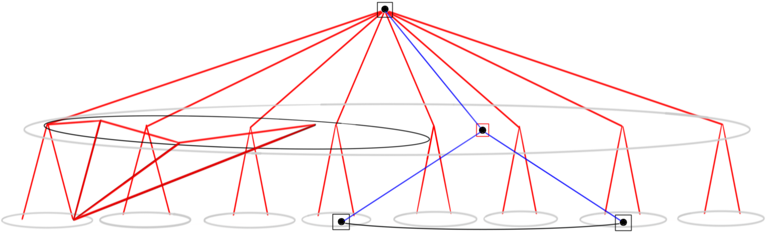

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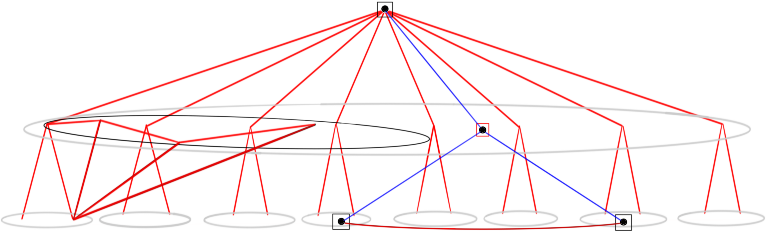

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3. Consider the property $\mathcal{B}_s$ that every pair of vertices have $s-2$ adjacent common neighbors.
   This property is increasing with sharp threshold probability $q(n) = (2(s-2)!)^{\frac{2}{(s+1)(s-2)}} n^{-\frac{2}{s+1}} (\ln n)^{\frac{2}{(s-2)(s+1)}}$.

   Consider the property $\mathcal{A}_s$ of having w-sat equal to w-sat(G_n, K_s).
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4. It remains to prove that, for $n^{-\frac{2}{s+1}} \leq p \leq (1+\varepsilon)q(n)$ the property $\mathcal{A}_s$ does not hold a.a.s.

Bipartite patterns

Simple exercise:

- w-sat$(K_n, K_{1,t}) = \binom{t}{2}$.

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Theorem (Kronenberg, Martins, Morrison, 2020)

- *w-sat*$(K_n, K_{t,t}) = (t-1)(n+1-t/2)$ *for* $n \geq 3t-3$*;*
- *w-sat*$(K_n, K_{t,t+1}) = (t-1)(n+1-t/2)+1$ *for* $n \geq 3t-3$*;*
- *w-sat*$(K_n, K_{t,s}) = (t-1)n + f(s,t)$ *for* $n \geq 4t$*, where* $f(s,t)$ *does not depend on* n*.*

Stability for stars

Theorem (Kalinichenko, Z, 2020++)

There is a constant C *such that*

- *if* $p \geq C(1+\varepsilon)(n \ln^{s-2} n)^{-1/(s-1)}$,

  then whp $w\text{-}sat(G(n,p), K_{1,t}) = \binom{t}{2}$;
- *if* $p \leq C(1-\varepsilon)(n \ln^{s-2} n)^{-1/(s-1)}$,

  then whp $w\text{-}sat(G(n,p), K_{1,t}) \neq \binom{t}{2}$.

Stability for bipartite graphs

Theorem (Kalinichenko, Z, 2020++)

Let p do not depend on n. Whp

- $w\text{-}sat(G(n,p), K_{t,t}) = (t-1)(n+1-t/2)$;
- $w\text{-}sat(K_n, K_{t,t+1}) = (t-1)(n+1-t/2)+1$;
- $\left| w\text{-}sat(G(n,p), K_{t,s}) - (t-1)n \right| \leq f(s,t)$,

where $f(s,t)$ does not depend on n.

Transferring lemma

Lemma

Let

- $p \in (0,1)$ *do not depend on* n,
- F *have minimum degree* $\delta \geq 1$,
- H_n *be weakly* F*-saturated in* K_n *on* $[n]$ *and have smallest possible number of edges,*
- *there exist* $m \in \mathbb{N}$ *such that, for every* n *large enough,*

  every vertex of H_n *outside* $[m]$ *have no neighbors outside* $[m]$ *and have exactly* $\delta - 1$ *neighbors inside* $[m]$.

Then, whp $\text{w-sat}(G(n,p), F) = \text{w-sat}(K_n, F)$.