Partition regularity for systems of Diophantine equations

Jonathan Chapman

The University of Manchester

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Partition regularity for linear equations

Consider a finite colouring of the positive integers

$$\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r.$$

What monochromatic structures are guaranteed to exist?

Theorem (Schur, 1916)

For some i, there exist $x, y, z \in C_i$ such that $x + y = z$.

Theorem (Van der Waerden, 1927)

For some i, the set C_i contains arbitrarily long arithmetic progressions $\{x, x+d, \ldots, x+(L-1)d\} \quad (d > 0)$.

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One can further show that there exists a colour class C_i which simultaneously satisfies the conclusions of both Schur and van der Waerden's theorems. We leave this as an exercise for the interested reader.

Partition regularity for linear equations

Definition (Partition regularity)

A system of equations is called *partition regular* if, for every finite colouring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$, there exists a solution $(x_1, \ldots, x_s) \in {C_i}^s$ for some $i \in \{1, \ldots, r\}$.

A foundational result of arithmetic Ramsey theory is Rado's criterion for partition regularity of linear systems.

Theorem (Rado, 1933)

Let $a_1, \ldots, a_s \in \mathbb{Z} \setminus \{0\}$. The equation

$$a_1 x_1 + \cdots + a_s x_s = 0$$

is partition regular if and only if there exists some non-empty set $I \subseteq \{1, \ldots, s\}$ such that $\sum_{i \in I} a_i = 0$.

This latter condition is equivalent to the assertion that the equation has a non-zero solution over $\{0, 1\}^s$.

Examples of equations which are not partition regular

Example 1

Consider the equation

$$y = 2x.$$

Let C_1 (respectively C_2) denote the set of positive integers with an odd (respectively even) number of 2's in their prime factorisation. Under this colouring, there are no monochromatic solutions to the above equation.

Example 2

Rado's theorem implies that the equation

$$x + y = 4z$$

is not partition regular. An example of a 3-colouring with no monochromatic solutions to this equation is given by

$$\mathbb{N} = \{1\} \cup \{2, 3\} \cup \{4, 5, 6, 7\} \cup \{8, \ldots, 15\} \cup \{16, \ldots, 31\} \cup \cdots .$$

That is, $x \in C_i$ if and only if $2^n \leqslant x < 2^{n+1}$ with $n \equiv i \pmod 3$.

Non-linear open problems (squares)

Conjecture (Erdős-Graham, 1980)

The Pythagorean equation $x^2 + y^2 = z^2$ is partition regular.

Conjecture (Gyarmati-Ruzsa, 2012)

The equation $x^2 + y^2 = 2z^2$ is non-trivially[1] partition regular. (Equivalently, every finite colouring of the squares produces monochromatic 3APs).

Conjecture (Folklore)

Let $a_1, \ldots, a_s \in \mathbb{Z} \setminus \{0\}$. The diagonal quadric

$$a_1 x_1^2 + \cdots + a_s x_s^2 = 0$$

is non-trivially partition regular if and only if $s \geqslant 3$ and there exists some non-empty $I \subseteq \{1, \ldots, s\}$ such that $\sum_{i \in I} a_i = 0$.

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[1] i.e. there exists a monochromatic solution $(x_1, \ldots, x_s)$ with x_i all distinct.

Recent progress

Theorem (Bergelson, 1996)

The equation $x - y = Mz^2$ *is partition regular for all* $\lambda \in \mathbb{N}$.

Theorem (Csikvári-Gyarmati-Sárközy, 2012)

The equation $x + y = z^2$ *is* **not** *non-trivially partition regular.*

Theorem (Moreira, 2017)

The equation $x^2 - y^2 = Mz$ *is partition regular for all* $\lambda \in \mathbb{N}$.

Theorem (Chow-Lindqvist-Prendiville, 2018)

The equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2$ *is partition regular.*

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Theorem (Heule-Kullmann-Marek[2], 2016)

The largest N *for which there exists a two colouring* $\{1, \dots, N\} = C_1 \cup C_2$ *with no monochromatic solution to* $x^2 + y^2 = z^2$ *is* $N = 7824$.

[2] Computer assisted proof of size ≈200Tb!

Non-linear open problems (higher powers)

Formulating similar conjectures for kth power equations with $k \geqslant 3$ is complicated by number theoretic obstructions. This is because it is generally very difficult to determine whether a Diophantine equation has non-trivial solutions.

Example (Wiles, 1995)

The equation $x^k + y^k = z^k$ obeys Rado's condition, however, for all $k \geqslant 3$, this equation has no solutions in positive integers.

Problem

For $k \geqslant 3$, determine sufficient conditions for an equation of the form

$$a_1 x_1^k + \cdots + a_s x_s^k = 0$$

to be partition regular.

Rado's criterion for squares and higher powers

By using tools from analytic number theory and harmonic analysis (in particular, the Hardy-Littlewood circle method) Chow, Lindqvist, and Prendiville obtained necessary and sufficient conditions for partition regularity for kth power equations in sufficiently many variables.

Theorem (Chow-Lindqvist-Prendiville, 2018)

Let $k \in \mathbb{N}$. There exists a quantity $s_0(k)$ such that the following is true for all $s \geqslant s_0(k)$. The diagonal equation

$$a_1 x_1^k + \cdots + a_s x_s^k = 0$$

is non-trivially partition regular if and only if there exists some non-empty $I \subseteq \{1, \ldots, s\}$ such that $\sum_{i \in I} a_i = 0$.

One may take $s_0(1) = 3$, $s_0(2) = 5$, $s_0(3) = 8$, and in general

$$s_0(k) = k(\log k + \log\log k + 2 + O(\log\log k / \log k)).$$

Rado's criterion for systems

Consider a system of n homogeneous linear equations in s variables:

$$\begin{aligned} a_{1,1}x_1 + \cdots + a_{1,s}x_s &= 0; \\ &\vdots \\ a_{n,1}x_1 + \cdots + a_{n,s}x_s &= 0. \end{aligned}$$

Observe that if this system is PR, then so is any equation which can be formed by taking linear combinations of these equations.

Example

Consider the pair of equations

$$\begin{aligned} x + y &= z; \\ x + y &= 2w. \end{aligned}$$

Individually, each of these two equations is partition regular. However, this system is **not** partition regular since solutions to this system must satisfy $z = 2w$, which is not partition regular.

Rado's criterion for systems

Rado showed that checking partition regularity for systems of equations can be reduced to verifying partition regularity for all linear combinations.

Theorem (Rado, 1933)

Let $\mathbf{A} = (a_{i,j})$ *be a non-zero* $n \times s$ *integer matrix. The system*

$$\begin{aligned} a_{1,1}x_1 + \cdots + a_{1,s}x_s &= 0; \\ &\vdots \\ a_{n,1}x_1 + \cdots + a_{n,s}x_s &= 0 \end{aligned}$$

is partition regular if and only if every equation which can formed by taking linear combinations of these equations is partition regular (or, equivalently, every equation defined by a non-zero vector from the row space of $\mathbf{A}$ *is partition regular).*

Partition regularity for non-linear systems

We now turn to the subject of generalising Rado's theorem to systems of kth power equations

$$\begin{aligned} a_{1,1}x_1^k + \cdots + a_{1,s}x_s^k &= 0; \\ &\vdots \\ a_{n,1}x_1^k + \cdots + a_{n,s}x_s^k &= 0. \end{aligned}$$

As with linear systems, a necessary condition for partition regularity is that every linear combination of these equations is partition regular. Furthermore, it is necessary that the underlying linear system (replacing each x_i^k with x_i) is partition regular.

Partition regularity for non-linear systems

However, these conditions are not sufficient in general.

Example (Failure of van der Waerden for higher powers)

Let $n \geqslant 3$. Consider the system of equations

$$x_1^k + x_3^k = 2x_2^k;$$

$$\vdots$$

$$x_{n-2}^k + x_n^k = 2x_{n-1}^k.$$

Note that $(x_1^k, \ldots, x_n^k)$ is an arithmetic progression of length n in kth powers. However, it was proved by Euler that there are no non-trivial 4APs in squares, and so this system is not non-trivially partition regular for $k = 2$ and $n \geqslant 4$.

Furthermore, it was shown by Darmon and Merel (1997) that there are no non-trivial 3APs in kth powers for $k \geqslant 3$, and so this system is also not non-trivially partition regular for $k \geqslant 3$ and $n \geqslant 3$.

Partition regularity for non-linear systems

However, these conditions are not sufficient in general.

Example (No Brauer configurations in higher powers)

Consider the pair of equations

$$x^2 + z^2 = 2y^2;$$
$$x^2 + t^2 = y^2.$$

Note that (x^2, y^2, z^2) is a 3AP with common difference t^2. However, a classical result of Fermat proves that every non-trivial 3AP in squares has non-square common difference. Thus, this system has no solutions over $\mathbb{N}$, and is therefore not partition regular.

Moreover, by the previous slide, the corresponding pair of equations for kth powers with $k \geqslant 3$ is also not partition regular.

Rado's criterion for non-linear systems

Assuming that the coefficient matrix is suitably non-singular, we can extend the results of Chow-Lindqvist-Prendiville to systems.

Theorem (C., 2020+)

Let $k \in \mathbb{N}$, and let $\mathbf{A} = (a_{i,j})$ be a non-zero $n \times s$ integer matrix. Suppose that the following condition holds:

(∗) *Given any $d \in \{1, \ldots, n\}$ and linearly independent vectors $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(d)}$ in the row space of $\mathbf{A}$, the $d \times s$ matrix with rows $\mathbf{v}^{(i)}$ has at least $dk^2 + 1$ non-zero columns.*

Then the system of equations

$$\begin{aligned} a_{1,1}x_1^k + \cdots + a_{1,s}x_s^k &= 0; \\ &\vdots \\ a_{n,1}x_1^k + \cdots + a_{n,s}x_s^k &= 0 \end{aligned}$$

is partition regular if and only if the underlying linear system $\mathbf{Ax} = \mathbf{0}$ is partition regular.

Density regularity

One can also ask whether arithmetic structures are guaranteed to exist inside all 'sufficiently large' sets. This is formalised as follows:

Definition (Density regularity)

A system of equations is called *density regular* if it has a non-trivial solution over every set $A \subseteq \mathbb{N}$ with *positive upper density*, meaning that

$$\limsup_{N \to \infty} \frac{|A \cap \{1, 2, \ldots, N\}|}{N} > 0.$$

Note that, by considering the largest colour class C_i, density regularity implies partition regularity.

Theorem (Szemerédi, 1975)

Let $\mathbf{A} = (a_{i,j}) \in \mathbb{Z}^{n \times s}$. The system of equations $\mathbf{A}\mathbf{x} = \mathbf{0}$ is density regular if and only if this system admits constant non-zero solutions or, equivalently, the columns of $\mathbf{A}$ sum to $\mathbf{0}$.

Density regularity for non-linear equations

As with partition regularity, there has been great interest in density regularity for non-linear systems.

Theorem (Browning-Prendiville, 2017)

For all $s \geqslant 5$ *and* $a_1, \ldots, a_s \in \mathbb{Z} \setminus \{0\}$, *the equation*

$$a_1 x_1^2 + \cdots + a_s x_s^2 = 0$$

is density regular if and only if $a_1 + \cdots + a_s = 0$.

Theorem (Chow-Lindqvist-Prendiville, 2018)

Let $k \in \mathbb{N}$. *For all* $s \geqslant k^2 + 1$, *the equation*

$$a_1 x_1^k + \cdots + a_s x_s^k = 0$$

is non-trivially density regular if and only if $a_1 + \cdots + a_s = 0$.

Density regularity for non-linear systems

Our methods allow us to generalise these two theorems to systems of suitably non-singular diagonal equations.

Theorem (C., 2020+)

Let $k \in \mathbb{N}$, and let $\mathbf{A} = (a_{i,j})$ be a non-zero $n \times s$ integer matrix. Suppose that the following condition holds:

(∗) *Given any $d \in \{1, \ldots, n\}$ and linearly independent vectors $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(d)}$ in the row space of $\mathbf{A}$, the $d \times s$ matrix with rows $\mathbf{v}^{(i)}$ has at least $dk^2 + 1$ non-zero columns.*

Then the system of equations

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is density regular if and only if the columns of $\mathbf{A}$ sum to $\mathbf{0}$.

Further reading

- **J. Chapman**, *Partition regularity for systems of diagonal equations*, preprint, arXiv:2003.10977.
- S. Chow, S. Lindqvist, and S. Prendiville, *Rado's criterion over squares and higher powers*, Journal of the European Mathematical Society, to appear. arXiv:1806.05002.

Thank you for your attention.