Improved bound on r-distant strong chromatic index

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The original problem

Adjacent vertex distinguishing index / Strong chromatic index

1. $G = (V, E)$ - a simple graph, no isolated edges, $\Delta(G) = \Delta$,
2. $c\colon E \to C$ - a proper colouring,
3. $S_c(v) = \{c(uv)\colon uv \in E\}$, $v \in V$,
4. $S_c(u) \neq S_c(v)$ for each $uv \in E$.

The minimum number k so that we are able to satisfy the above conditions is denoted by $\chi'_a(G)$.

Conjecture (Zhang,Liu,Wang 2002)

For every connected graph G other than K_2 or C_5,

$$\chi'_a(G) \leq \Delta(G) + 2.$$

Results on $\chi'_a(G)$

Theorem [Hatami 2005]

If G is a graph with no isolated edges and maximum degree $\Delta > 10^{20}$, then

$$\chi'_a(G) \leq \Delta + 300.$$

Theorem [Joret, Lochet 2020+]

If G is a graph with no isolated edges and large enough maximum degree Δ, then

$$\chi'_a(G) \leq \Delta + 19.$$

A side problem

Adjacent vertex distinguishing edge choice number

1. $G = (V, E)$ - a simple graph, no isolated edges, $\Delta(G) = \Delta$,
2. $c \colon E \to C$,
3. $S_c(v) = \{c(uv) \colon uv \in E\}$, $v \in V$,
4. For each edge $e \in E$ a set L_e is given,
5. The colouring needs to be proper, with $c(e) \in L_e$, $e \in E$ and $S_c(u) \neq S_c(v)$, $uv \in E$.

The minimum number k so that for every set of lists $\{L_e\}_{e \in E}$ of size k we are able to satisfy the above conditions is denoted by $\mathrm{ch}'_a(G)$.

Conjecture (Horněk, Woźniak 2012)

For every connected graph G other than K_2 or C_5,

$$\mathrm{ch}'_a(G) = \chi'_a(G).$$

Results on $\text{ch}'_a(G)$

Theorem [Przybyło, Wong 2015]

If G is a graph with no isolated edges, maximum degree Δ and coloring number $\text{col}(G)$, then

$$\text{ch}'_a(G) \leq \text{ch}'_\Sigma(G) \leq \Delta + 3\text{col}(G) - 4,$$

where $\text{ch}'_\Sigma(G)$ is the neighbour sum distinguishing index of G, i.e. the parameter where we require distinctness of *sums* of labels at a vertex rather than sets.

Note

As AVD colouring from lists is the special case of an edge colouring from lists, then:

$$\text{ch}'_a(G) \geq \text{ch}'(G)$$

where $\text{ch}'(G)$ is a list chromatic index.

Results on $\mathrm{ch}'_a(G)$

Theorem [Molloy, Reed, 2000]

There is a constant k such that $\mathrm{ch}'(G) \leq \Delta(G) + k\Delta(G)^{1/2}(\log \Delta(G))^4$ for every graph G with maximum degree Δ.

Theorem [JK, Przybyło, 2018]

There is a constant C such that

$$\mathrm{ch}'_a(G) \leq \Delta + C\Delta^{1/2}(\log \Delta)^4$$

for every graph G with maximum degree Δ and without isolated edges.

The titular problem

r-distant strong chromatic index

1. $G = (V, E)$ - a simple graph, no isolated edges, $\Delta(G) = \Delta$,
2. $c \colon E \to C$ - a proper colouring,
3. $S_c(v) = \{c(uv) \colon uv \in E\}$, $v \in V$,
4. $S_c(u) \neq S_c(v)$ for each $u, v \in V$ such that $d(u, v) \leq r$.

The minimum number k so that we are able to satisfy the above conditions is denoted by $\chi'_{a,r}(G)$.

Conjecture (Przybyło 2018)

For each positive integer r there exist constants δ_0 and C such that

$$\chi'_{a,r}(G) \leq \Delta(G) + C$$

for every graph without an isolated edge and with $\delta(G) \geq \delta_0$.

Results on $\chi'_{a,r}(G)$

Theorem [Przybyło 2018]

For every positive $\varepsilon \leq 1$ and a positive integer r, there exists Δ_0 and a constant $C = C(\varepsilon, r)$ such that:

$$\chi'_{a,r}(G) \leq \Delta(G) + C$$

for every graph G without an isolated edge and with $\delta(G) \geq \varepsilon\Delta(G)$, $\Delta(G) \geq \Delta_0$. In particular, $C \leq \varepsilon^{-2}(7r + 200) + r + 6$.

Theorem [JK, Prorok 2020+]

For each positive integer r there exist Δ_0, δ_0 and C such that

$$\chi'_{a,r}(G) \leq \Delta(G) + C$$

for every graph G without an isolated edge and with $\Delta(G) \geq \Delta_0$ and $\delta(G) \geq \delta_0$.

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