Quasirandom-forcing Tournaments

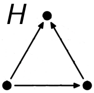
Fiona Skerman (Uppsala University)

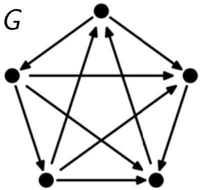
joint work with Robert Hancock, Adam Kabela, Dan Kráľ,
Taísa Martins, Roberto Parente and Jan Volec

We recall tournaments

Definitions

A *tournament* is a directed graph having precisely one arc between each pair of its nodes.





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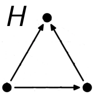

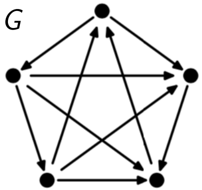

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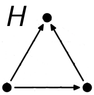
$d(H,G)$ = probability that $|H|$ randomly chosen vertices of G induce H

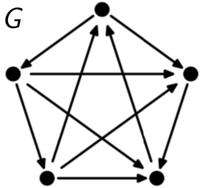
$d^*(H,G)$ = probability that an ordered set of $|H|$ randomly chosen vertices of G induces a labelled copy of H

$$d(H,G) = \frac{n(H,G)}{\binom{|G|}{|H|}} = \frac{8}{\binom{5}{3}}$$

$$d^*(H,G) = \frac{n^*(H,G)}{|G|_{|H|}} = \frac{8}{5.4.3}$$

$$d^*(H,G) = \frac{|\mathrm{Aut(H)}|}{|H|!} d(H,G)$$





Quasirandomness of tournaments

Definitions Chung-Graham '91 (Chung-Graph-Wilson '89, Thomason 87')
Let (G_n) be sequence of tournaments (such that $|G_n| \to \infty$ as $n \to \infty$).
We say that (G_n) is *quasirandom* if

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We say a set of tournaments $\mathcal{H}$ is *quasirandom-forcing* if

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Thm (Chung-Graham '91)

Let $h \geq 4$ and define $\mathcal{H}_h$ to be the set of tournaments on h nodes. Then $\mathcal{H}_h$ is quasirandom-forcing.

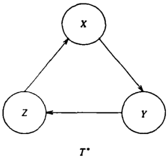

FIGURE 1
Chung-Graham '91

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A tournament H is *quasirandom-forcing* if

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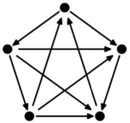
Which tournaments on h nodes are quasirandom-forcing?

- For $h \geq 4$, every transitive tournament is quasirandom-forcing (Coregliano-Razborov '17, Lovasz '93)
- For $h = 5$, F_5 is quasirandom-forcing, others not except perhaps H_5. (Coregliano, Parente and Sato '19).
- For $h \geq 7$, the only quasirandom-forcing is the transitive (Bucić, Long, Shapira and Sudakov, '20+). Also local-forcing.
- For $h = 6$, the only quasirandom-forcing is the transitive. H_5 not. (Hancock, K., Kráľ, Martins, Parente, Skerman and Volec, '20+).

F_5



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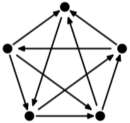

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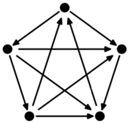


Summary: H is quasirandom-forcing iff H is F_5 or H is the transitive tournament on $h \geq 4$ nodes.

cf. Goodman '59, Beineke, Harary '65.

General Arguments

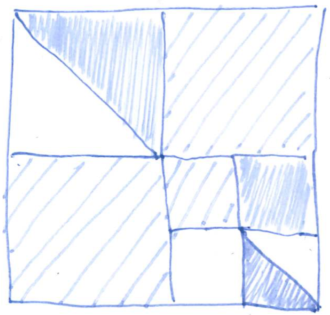
Proposition (Bucić, Long, Shapira and Sudakov '20+)

Let H be a non-transitive tournament on $h \geq 7$ nodes. Then H is not quasirandom-forcing.

Lemma (Hancock, Kabela, Král', Martins, Parente, S. and Volec '20+)

Let H be a non-transitive tournament on 6 nodes. If H **contains twins** or has a **non-trivial automorphism group** or is **not strongly connected** then H is is not quasirandom-forcing.

A *tournamenton* is measurable function W
$W : [0,1]^2 \to [0,1]$ with $W(x,y) = 1 - W(y,x)$, $\forall x, y$.



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W-random graph $S = \mathbb{G}(h, W)$.
sample $x_1, \ldots, x_h \in^u [0,1]$, for $i < j$:
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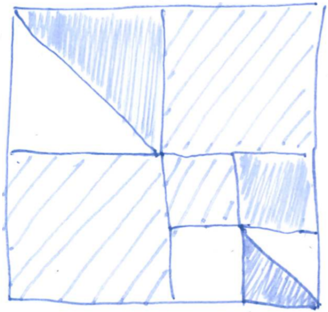

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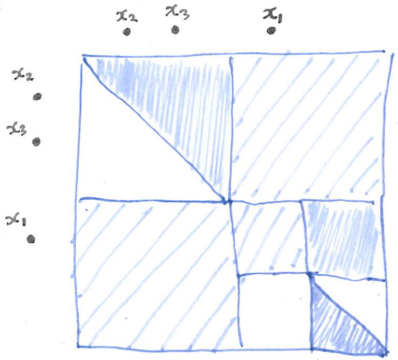

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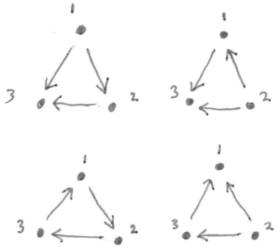

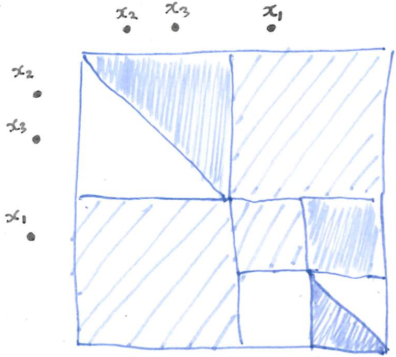

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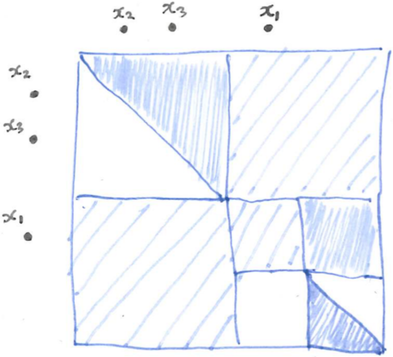

labelled density
$$d^*(H, W) = \mathbb{P}(S =^{\text{labelled}} H)$$
$$= \int_{[0,1]^h} \prod_{\overrightarrow{ij} \in E(H)} W(x_i, x_j)\, \mathrm{d}x_1 \cdots \mathrm{d}x_h.$$

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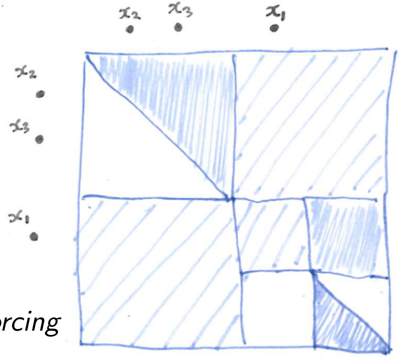

H is *quasirandom-forcing*
if $d^*(H, W) = 2^{-\binom{h}{2}}$ implies
$W = 1/2$ almost everywhere.

Proposition (BLSS, HKKMPSV):
Non-transitive H is *not quasirandom-forcing*
if $\exists W$ with $d^*(H, W) > 2^{-\binom{h}{2}}$.

General Arguments

Proposition (Bucić, Long, Shapira and Sudakov, 2020+)

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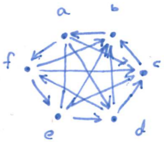

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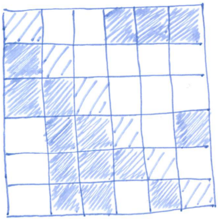
W_H



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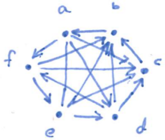
Let H be a non-transitive tournament on $h \geq 7$ nodes. Then H is not quasirandom-forcing.

Pf (BLSS): $d^*(H, W_H) \geq h^{-h}$. For $h \geq 7$, $h^{-h} \geq 2^{-\binom{h}{2}}$ $\square$.

$6^{-6} < 2^{-15} < 2 \times 6^{-6}$

H



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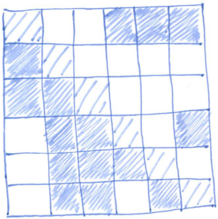


General Arguments

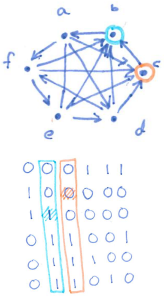
$55 + 1$

Lemma (Hancock, Kabela, Kráľ, Martins, Parente, S. and Volec, 2020+)

Let H be a non-transitive tournament on $h = 6$ nodes. If H **contains twins** then H is not quasirandom-forcing.

Let $N^+(x)$ denote out-neighbours of x.
Nodes u and v are *twins* if $N^+(x)\backslash\{y\} = N^+(y)\backslash\{x\}$.

H



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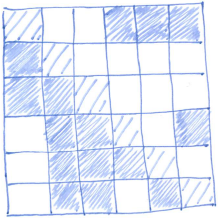

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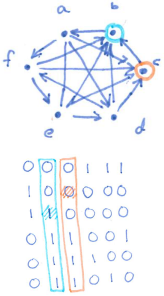

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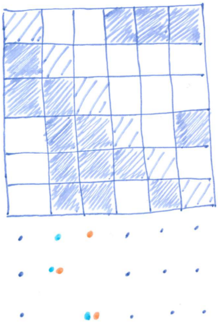

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Pf: $d^*(H, W_H) \geq h^{-h}(1 + \frac{1}{2} + \frac{1}{2})$. For $h = 6$, $2h^{-h} \geq 2^{-\binom{h}{2}}$ $\square$.

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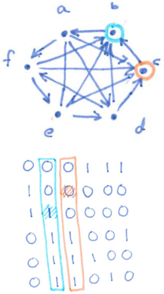

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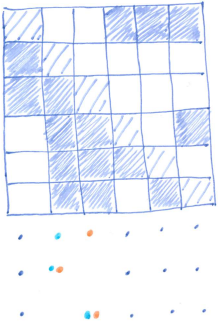

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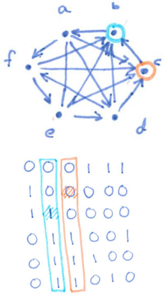

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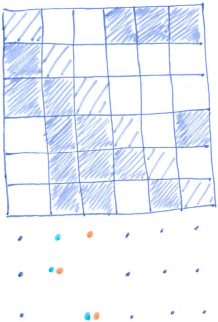

General Arguments

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Let H be a non-transitive tournament on $h = 6$ nodes. If H has a **non-trivial automorphism** then H is not quasirandom-forcing.

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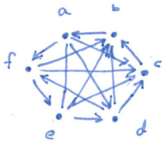

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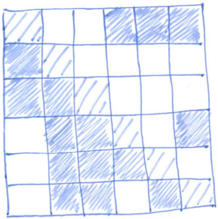

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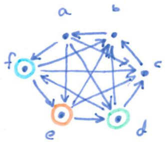

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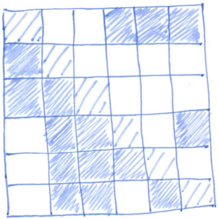

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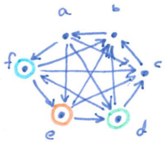

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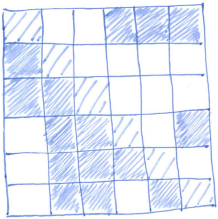

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a b c e f d

a b c f d e

General Arguments $18+1$

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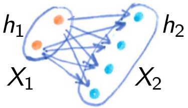
Let H be a non-transitive tournament on $h \geq 4$ nodes.
If H is **not strongly connected** then H is not quasirandom-forcing.

H



W_α

General Arguments

18 + 1

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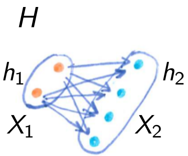
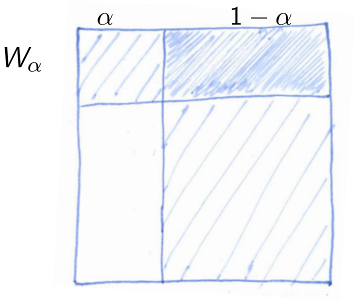




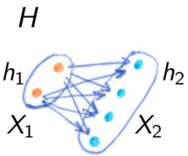
General Arguments

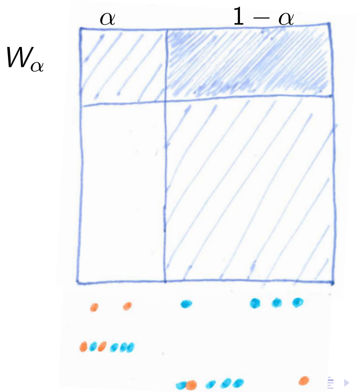
18 + 1

Lemma (Hancock, Kabela, Kráľ, Martins, Parente, S. and Volec, 2020+)

Let H be a non-transitive tournament on $h \geq 4$ nodes.
If H is **not strongly connected** then H is not quasirandom-forcing.

Pf: $d^*(H, W_\alpha) \geq \alpha^{h_1}(1-\alpha)^{h_2} 2^{-\binom{h_1}{2}-\binom{h_2}{2}} + \alpha^h 2^{-\binom{h}{2}} + (1-\alpha)^h 2^{-\binom{h}{2}}$.





General Arguments $\mathbf{14+1}$

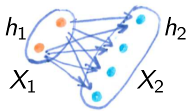
Lemma (Hancock, Kabela, Kráľ, Martins, Parente, S. and Volec, 2020+)

Let H be a non-transitive tournament on $h \geq 4$ nodes.
If H is **not strongly connected** then H is not quasirandom-forcing.

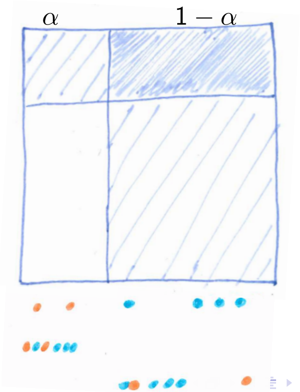
Pf: $d^*(H, W_\alpha) \geq \alpha^{h_1}(1-\alpha)^{h_2} 2^{-\binom{h_1}{2}-\binom{h_2}{2}} + \alpha^h 2^{-\binom{h}{2}} + (1-\alpha)^h 2^{-\binom{h}{2}}$.

$\Rightarrow \exists \alpha,\ d^*(H, W_\alpha) > 2^{-\binom{h}{2}}$ $\square$.

H



W_α

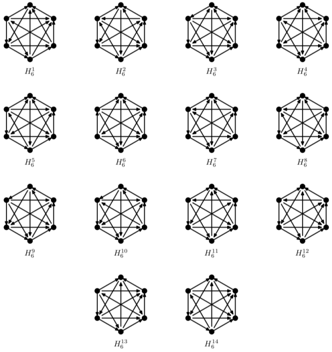


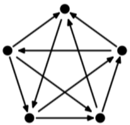
General Arguments

$14 + 1$

Lemma (Hancock, Kabela, Král', Martins, Parente, S. and Volec '20+)

Let H be a non-transitive tournament on 6 nodes. If H **contains twins** or has a **non-trivial automorphism group** or is **not strongly connected** then H is is not quasirandom-forcing.





General Arguments

$14 + 1$

(A) not strongly connected

(B) non-trivial automorphism group

(C) contains twins

A	B	C	D	E	Tournament	
•	•	•			$00000, 0000, 000, 01, 0$	
•		•			$00010, 0000, 000, 00, 0$	
		•			$00011, 0000, 000, 00, 0$	
		•			$00010, 0001, 000, 00, 0$	
				•	$00010, 0000, 001, 00, 0$	H_6^1
		•			$00010, 0000, 000, 01, 0$	
		•			$00010, 0000, 000, 00, 1$	
•		•			$00000, 0010, 000, 00, 0$	
		•			$00001, 0010, 000, 00, 0$	
•		•			$00000, 0011, 000, 00, 0$	
•					$00000, 0010, 001, 00, 0$	
•					$00000, 0010, 000, 01, 0$	
•		•			$00000, 0010, 000, 00, 1$	
•	•				$00000, 0011, 001, 00, 0$	
•	•	•			$00000, 0000, 010, 00, 0$	
	•				$00001, 0000, 010, 00, 0$	
•	•				$00000, 0001, 010, 00, 0$	
•		•			$00000, 0000, 011, 00, 0$	
•		•			$00100, 0000, 000, 00, 0$	
•		•			$00110, 0000, 000, 00, 0$	
		•			$00111, 0000, 000, 00, 0$	
		•			$00110, 0001, 000, 00, 0$	
		•			$00110, 0000, 001, 00, 0$	
		•			$00110, 0000, 000, 01, 0$	
		•			$00110, 0000, 000, 00, 1$	
		•			$00111, 0000, 001, 00, 0$	
	•	•			$00110, 0001, 001, 00, 0$	
	•	•			$00111, 0000, 000, 01, 0$	

A	B	C	D	E	Tournament	
			•		$00110, 0001, 000, 01, 0$	H_6^2
•					$00100, 0010, 000, 00, 0$	
			•		$00101, 0010, 000, 00, 0$	H_6^3
		•			$00100, 0011, 000, 00, 0$	
			•		$00100, 0010, 001, 00, 0$	H_6^4
			•		$00100, 0010, 000, 01, 0$	H_6^5
				•	$00100, 0010, 000, 00, 1$	H_6^6
	•				$00101, 0010, 001, 00, 0$	
				•	$00100, 0011, 001, 00, 0$	H_6^7
			•		$00100, 0011, 000, 01, 0$	H_6^8
•	•				$00110, 0010, 000, 00, 0$	
				•	$00111, 0010, 000, 00, 0$	H_6^9
		•			$00111, 0011, 000, 00, 0$	
			•		$00111, 0010, 001, 00, 0$	H_6^{10}
•	•	•			$00000, 0100, 000, 00, 0$	
•	•				$00010, 0100, 000, 00, 0$	
	•	•			$00011, 0100, 000, 00, 0$	
			•		$00010, 0101, 000, 00, 0$	H_6^{11}
	•				$00010, 0100, 000, 00, 1$	
•	•	•			$01000, 0000, 000, 00, 0$	
•	•				$01000, 0000, 000, 01, 0$	
•					$01010, 0000, 000, 00, 0$	
		•			$01011, 0000, 000, 00, 0$	
			•		$01010, 0001, 000, 00, 0$	H_6^{12}
			•		$01010, 0000, 001, 00, 0$	H_6^{13}
				•	$01010, 0000, 000, 01, 0$	H_6^{14}
		•			$01010, 0000, 000, 00, 1$	

Excluding the rest $14 + 1$

We readily check the properties from the previous slide and consider the 14 remaining tournaments on 6 nodes plus 1 tournament on 5 nodes.

To show tournament H on h nodes is not quasirandom-forcing

- find tournament T with many copies of H.
  $n(H,T) > |T|^h 2^{-\binom{h}{2}} \Rightarrow d^*(H, W_T) > 2^{-\binom{h}{2}}$
- find a step tournament by perturbing about $1/2$.
  $d^*(H, A_x) = f(x)$ and find x such that $f(x) > 2^{-\binom{h}{2}}$.

$$A_x = \begin{pmatrix} 1/2 & 1/2-x \\ 1/2+x & 1/2 \end{pmatrix},$$

$$B_x = \begin{pmatrix} 1/2 & 1/2-x & 1/2+x \\ 1/2+x & 1/2 & 1/2-x \\ 1/2-x & 1/2+x & 1/2 \end{pmatrix},$$

$$C_x = \begin{pmatrix} 1/2 & 1/2-x & 1/2+x & 1/2-x \\ 1/2+x & 1/2 & 1/2-x & 1/2-x \\ 1/2-x & 1/2+x & 1/2 & 1/2-x \\ 1/2+x & 1/2+x & 1/2+x & 1/2 \end{pmatrix}.$$

Excluding the rest

$9+1$

We readily check the properties from the previous slide and consider the 14 remaining tournaments on 6 nodes plus 1 tournament on 5 nodes.

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- find tournament T with many copies of H.
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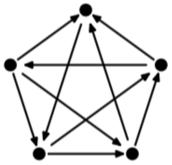
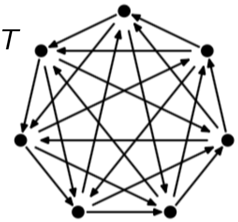
H_5



$n(H_5,T)=21$

T



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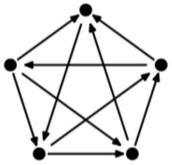
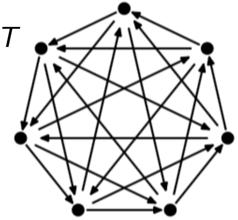
H_5

$n(H_5, T) = 21$



T



Thank you for your attention.

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