Modulo 2 and 4 behavior of values of certain partition functions

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Short plan of the presentation

- Introduction and basic identities.

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- Some congruences for $f_{2,5}(n)$ and $g_{2,5}(n)$.

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- Some congruences for $f_{2,5}(n)$ and $g_{2,5}(n)$.
- A general result.
- Some conjectures.

Let $A \subset \mathbb{N}_+$ be given and take $n \in \mathbb{N}$. By a partition of a non-negative integer n with parts in A, we mean any representation of n in the form

$$n = a_1 + \ldots + a_k, a_1 \geq a_2 \geq \ldots a_k$$

where $a_i \in A$. In particular, if $A = N_+$, then $p(n)$ is the famous partition function studied by Ramanujan. The function $p(n)$ counts the number of partitions with parts in $\mathbb{N}_+$, i.e., unrestricted partitions of n.

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Let $a, b \in \mathbb{N}_+$ are given and consider two partition functions. The function $f_{a,b}(n)$ counts the partitions of n into distinct parts no divisible by a or b; and the function $g_{a,b}(n)$ counts the partitions of n into parts not divisible by a or b.

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Using generating functions approach we easily obtain the identities

$$F_{a,b}(q) = \prod_{n=1}^{\infty} \frac{(1+q^n)(1+q^{[a,b]n})}{(1+q^{an})(1+q^{bn})} = \sum_{n=0}^{\infty} f_{a,b}(n)q^n,$$

$$G_{a,b}(q) = \prod_{n=1}^{\infty} \frac{(1-q^{an})(1-q^{bn})}{(1-q^n)(1-q^{[a,b]n})} = \sum_{n=0}^{\infty} g_{a,b}(n)q^n.$$

As usual, $[a, b]$ denotes the least common multiple of the integers a, b.

The functions $F_{a,b}$, $G_{a,b}$ are multiplicative inverse of each other when considered modulo 2. In other words

$$F_{a,b}(x)G_{a,b}(x) \equiv 1 \pmod{2}.$$

Equivalently, we can write

$$\sum_{i=0}^{n} f_{a,b}(i)g_{a,b}(n-i) \equiv 0 \pmod{2}, \quad n \in \mathbb{N}_{+}.$$

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There is a connection between $f_{2,b}$ and so called b-regular partitions. Let b be an odd positive integer and $(u_b(n))_{n\in\mathbb{N}}$ be the sequence counting b-regular partitions, i.e., $u_b(n)$ counts the number of partitions of n with parts not divisible by b. It is well know that

$$U_b(q) = \sum_{n=0}^{\infty} u_b(n)q^n = \prod_{n=1}^{\infty} \frac{1-q^{bn}}{1-q^n}.$$

On can prove that $f_{2,b}(n) \equiv u_b(n) \pmod{2}$ for all $n \in \mathbb{N}$.

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In the sequel, we are mainly interested in the case when $(a,b) = (2,5)$.

First, we recall the Jacobi triple product identity. More precisely, one among the many possible forms of this identity is the following

$$J(a,q) := \prod_{n=1}^{\infty}(1+a^{-1}q^{2n-1})(1+aq^{2n-1})(1-q^{2n}) = \sum_{n=-\infty}^{\infty} a^{n}q^{n^2}.$$

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We recall two more identities. The first one is called Euler's pentagonal number theorem:

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}}.$$

It can be easily deduced from the Jacobi triple identity by taking the substitution $q \to q^{3/2}$ and $a \to -q^{1/2}$.

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The second one is called Jacobi's identity:

$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}}.$$

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Theorem 1

For each $n \in \mathbb{N}$ the following congruences are true

$$f_{2,5}(4n+2) \equiv f_{2,5}(10n+2) \equiv f_{2,5}(10n+6) \equiv 0 \pmod{2},$$
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Proof: From the Jacobi's triple product identity we have

$$\begin{aligned} F_{2,5}(q)\prod_{n=1}^{\infty}(1-q^{10n})^2 &= J(q^2,q^5)J(q^4,q^5) \\ &= \left(\sum_{n=-\infty}^{\infty} q^{5n^2+2n}\right)\left(\sum_{n=-\infty}^{\infty} q^{5n^2+4n}\right) \qquad (1) \\ &= \sum_{x,y=-\infty}^{\infty} q^{5x^2+2x+5y^2+4y} = \sum_{n=0}^{\infty} C_{2,5}(n)q^n, \end{aligned}$$

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where

$$C_{2,5}(n) = \#\{(x,y)\in\mathbb{Z}\times\mathbb{Z}:\ 5x^2+2x+5y^2+4y=n\}.$$

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Consequently, by comparison of coefficients on both sides of the first and the last expression in (1) we get (modulo 2) recurrence relation

$$f_{2,5}(0)=1,\quad f_{2,5}(n)\equiv C_{2,5}(n)-\sum_{k=1}^{\lfloor\frac{n}{20}\rfloor}a_kf_{2,5}(n-20k) \pmod{2}. \qquad (2)$$

We prove that $f_{2,5}(10n+2) \equiv 0 \pmod{2}$ for each $n \in \mathbb{N}$. Using (2) we get the recurrence relation

$$f_{2,5}(2) = 0, \quad f_{2,5}(10n+2) \equiv C_{2,5}(10n+2) - \sum_{k=1}^{\lfloor \frac{5n+1}{10} \rfloor} a_k f_{2,5}(10(n-2k)+2) \pmod{2}.$$

We need to show that $C_{2,5}(10n+2)$ is even.

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To do that we consider the affine map $L : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q} \times \mathbb{Q}$ defined by

$$L(x, y) = \left(\frac{1}{5}(4x + 3y + 1), \frac{1}{5}(3x - 4y - 3) \right).$$

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The map L has the following properties:

(1) $Q \circ L = Q$;
(2) the map L has order 2, i.e., $L \circ L = \mathrm{Id}$, where $\mathrm{Id}(x,y) = (x,y)$ is the identity map on $\mathbb{Q} \times \mathbb{Q}$;
(3) there is no $n \in \mathbb{N}$ and $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ satisfying $10n+2 = Q(x,y)$ and $L(x,y) = (x,y)$;
(4) if $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ and $10n+2 = Q(x,y)$ for some $n \in \mathbb{N}$, then $L(x,y) \in \mathbb{Z} \times \mathbb{Z}$.

Thus, L is a well defined map from the set of integers solutions of $Q(x, y) = 10n + 2$ to itself. Moreover, the above reasoning shows that each integer solution of the equation $Q(x, y) = 10n + 2$ has the unique (integer) partner $L(x, y)$. Thus $C_{2,5}(10n + 2) \equiv 0 \pmod{2}$.

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We note that the first 10 terms of the sequence of interest are the following:

$$(f_{2,5}(10n + 2))_{n\in\mathbb{N}} = (0, 2, 4, 10, 20, 42, 76, 142, 244, 420, \dots),$$

and are even. To finish the proof we apply induction and we are done.

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The congruence $f_{2,5}(10n+6) \equiv 0 \pmod{2}$ can be proved in a similar way. First we need to prove that $C_{2,5}(10n+6) \equiv 0 \pmod{2}$. This can be done with the help of the linear map

$$L'(x,y) = \left(-\frac{1}{5}(4x+3y+3), -\frac{1}{5}(3x-4y+1)\right)$$

and exactly the same reasoning as in the case of the map L above. Using then the relation (2) and induction we easily get the result.

To get $f_{2,5}(20n+2) \equiv 0 \pmod{4}$ we back to identity (1) and obtain

$$\begin{aligned} f_{2,5}(20n+2) &= -\sum_{k=0}^{2n} b_k f_{2,5}(10(2n-k)+2) \\ &= -\sum_{k=1}^{n} b_{2k} f_{2,5}(20(n-k)+2) - \sum_{k=0}^{n-1} b_{2k+1} f_{2,5}(10(2n-2k-1)+2). \end{aligned} \tag{3}$$

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Exactly the same technique as above can be used in the proof of the congruence $f_{2,5}(20n+6) \equiv 0 \pmod{4}$. □

Corollary 2

The following inequalities are true:

$$\limsup_{N\to+\infty} \frac{\#\{n \le N : \; f_{2,5}(n) \equiv 0 \pmod{4}\}}{N} \ge \frac{1}{10}.$$

Using similar techniques one can prove the following

Theorem 3

For each $n \in \mathbb{N}$ the following congruences are true

$$g_{2,5}(4n+3) \equiv g_{2,5}(5n+3) \equiv g_{2,5}(5n+4) \equiv 0 \pmod{2}.$$

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Consequently, the following density result is true

Corollary 4

The following inequality is true:

$$\limsup_{N\to+\infty} \frac{\#\{n \leq N :\ g_{2,5}(n) \equiv 0 \pmod{2}\}}{N} \geq \frac{3}{5}.$$

A general result

Let $u, v_1, v_2 \in \mathbb{N}_+$ and suppose that $u > \max\{v_1, v_2\}$. We define

$$H_{u,v_1,v_2}(q) = \prod_{n=1}^{\infty}(1+q^{2un-u-v_1})(1+q^{2un-u+v_1})(1+q^{2un-u-v_2})(1+q^{2un-u+v_2})$$
$$= \sum_{n=0}^{\infty} h_{u,v_1,v_2}(n)q^n,$$

i.e., the number $h_{u,v_1,v_2}(n)$ counts the partitions of n into distinct parts which are congruent to $u \pm v_1, u \pm v_2 \pmod{2u}$.

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Theorem 5

Let $s \geq 2$ and $u = 2s+1, v_1 = 2r, v_2 = 2r-2$ for $s \geq r \geq 2$, then the following congruence is true $h_{u,v_1,v_2}(4n+2) \equiv 0 \pmod 2$

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The proof of this result uses the recurrence

$$h_{u,v_1,v_2}(n) \equiv C_{u,v_1,v_2}(n) - \sum_{k=1}^{\lfloor \frac{n}{4u} \rfloor} a_k h_{u,v_1,v_2}(n-4uk) \pmod{2},$$

where $C_{u,v_1,v_2}(n) = \#\{(x,y) \in \mathbb{Z} \times \mathbb{Z} : \ ux^2 + v_1 x + uy^2 + v_2 y = n\}$.

Some conjectures

Conjecture 1

We have the following:

$$g_{2,5}(2n+1) \equiv 1 \pmod{2} \iff n = 5m(3m-1) \text{ for some } m \in \mathbb{Z}.$$

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Conjecture 2

For each $n \in \mathbb{N}$ the following congruences are true

$$g_{2,5}(20n+7) \equiv g_{2,5}(20n+15) \equiv 0 \pmod{4}.$$

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A natural question arises whether we can expect further congruences modulo higher powers of 2. Our numerical search suggest the following.

Conjecture 3

For each $n \in \mathbb{N}$ the following congruences are true

$$u_5(10n) \equiv f_{2,5}(10n) \pmod{4},$$
$$u_5(20n+1) \equiv f_{2,5}(20n+1) \pmod{4},$$
$$u_5(20n+18) \equiv f_{2,5}(20n+18) \pmod{4}.$$

Interestingly, we were unable to spot similar congruences in case of $b > 5$.

Thank you for your attention;-)