COMPLEXITY OF H-COLORING IN HEREDITARY GRAPH CLASSES

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8th PCC, 14-18.09.2020

GRAPH HOMOMORPHISMS

- we have two graphs : G and H

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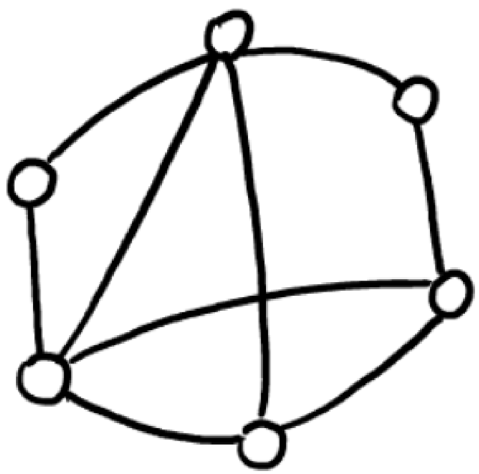

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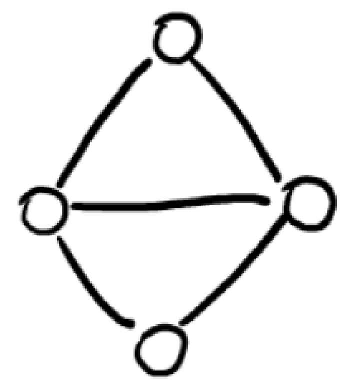

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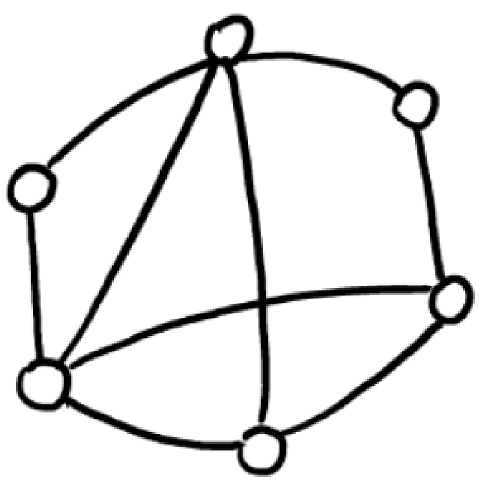

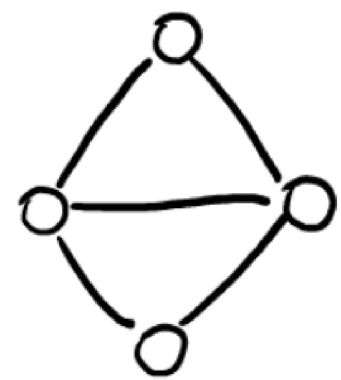

a **homomorphism** from G to H :

$$\boxed{f : V(G) \to V(H) \text{ s.t. } uv \in E(G) \Rightarrow f(u)\,f(v) \in E(H)}$$

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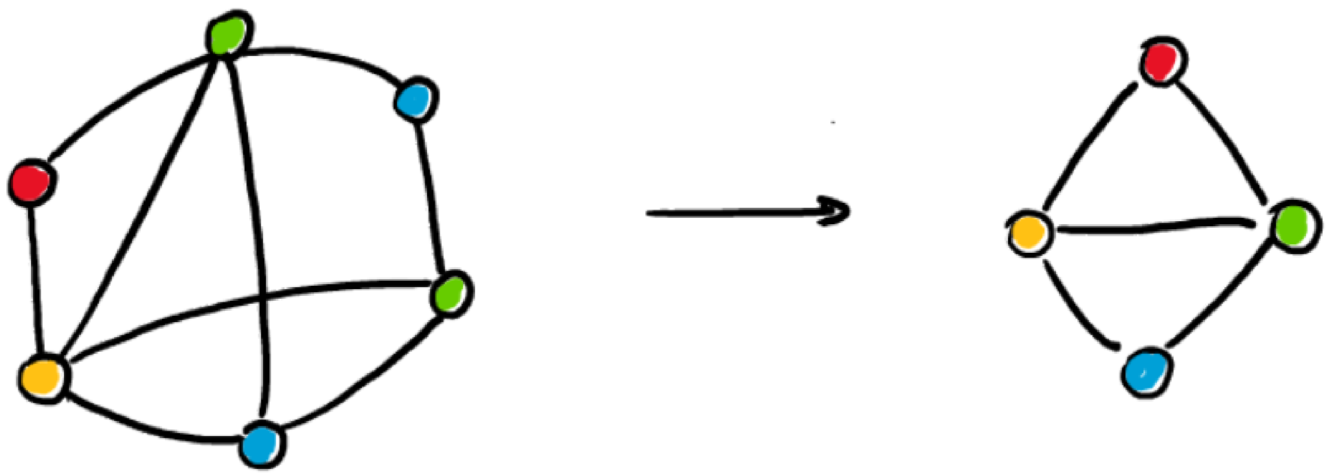

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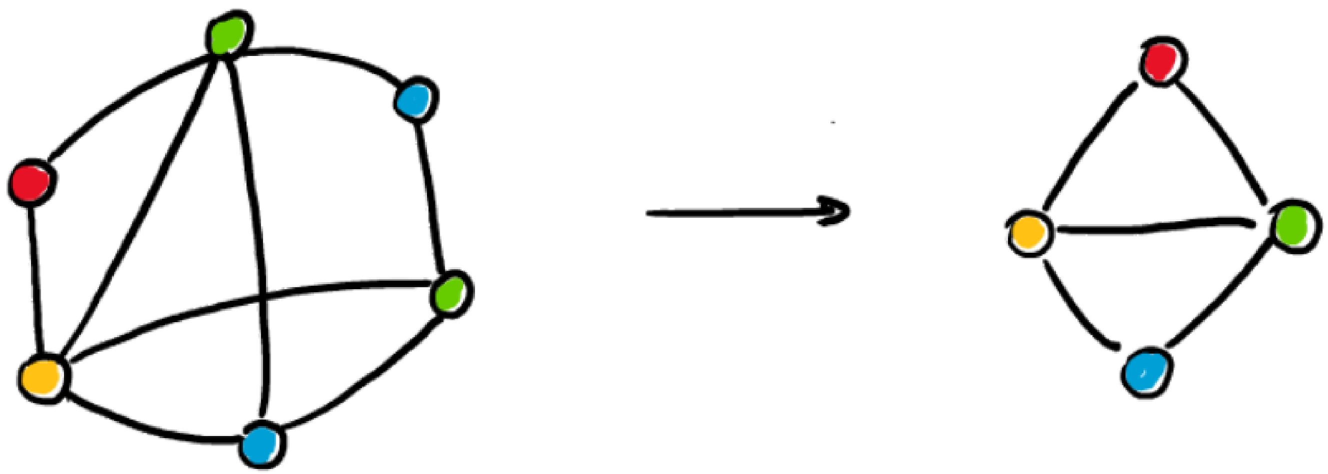

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a homomorphism from G to H : (informally)

V(H) are colors
V(G) are colored by V(H)
in a way that colors can be adjacent in G only if they are adjacent in H

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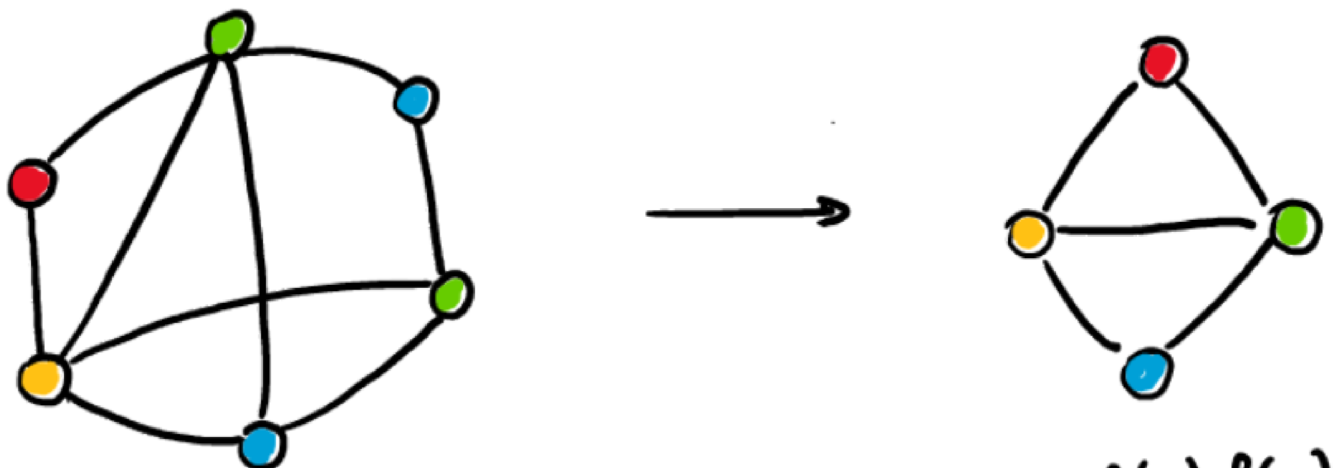

(colors indicate the function f)

$f: V(G) \to V(H)$ s.t. $uv \in E(G) \Rightarrow f(u)f(v) \in E(H)$

- no loops in $H \Rightarrow f$ is a proper $|H|$-coloring of G (possibly with additional restrictions, e.g., no red-blue edge)
- H is a k-clique $\Leftrightarrow f$ is a proper k-coloring of G
- $G \to H$: G admits a homomorphism to H

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denote the problem by $\mathrm{HOM}(H)$

- $\mathrm{Hom}(K_k)$ problem $\equiv$ k-COLORING problem

GRAPH HOMOMORPHISM PROBLEM

- if H is bipartite or has a loop, we can solve HOM(H) in polynomial time (w.r.t. $|V(G)|$)
- otherwise, HOM(H) is NP-complete [1] & no subexponential-time algorithm assuming ETH [2]

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subexponential time

e.g., $2^{O(\sqrt{n})}$, $2^{O(n^{0.9})}$ algorithms for instances on n vertices

VS

no subexponential time

no $2^{o(n)}$ algorithm assuming the Exponential Time Hypothesis (ETH-hard)

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- LHOM(K_k) $\equiv$ LIST k-COLORING

LIST HOMOMORPHISM PROBLEM

a generalization of HOM(H)
denoted by LHOM(H)

- if H is bi-arc, then LHOM(H) polynomial-time solvable
- otherwise, LHOM(H) NP-complete

& no subexponential algorithm assuming ETH [3]

PROBLEM

$\mathcal{C}$ - some graph class

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Can we solve Hom(H) | LHom(H) faster assuming our instances are from $\mathcal{C}$?

What faster can mean?

- classic approach

  polynomial-time solvable vs NP-complete

- fine-grained approach

  subexponential-time solvable vs ETH-hard

FORBIDDEN STRUCTURES

F - fixed connected graph

Graph G is F-free if it contains no F
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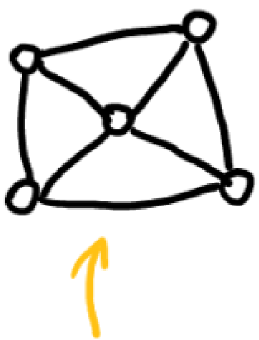
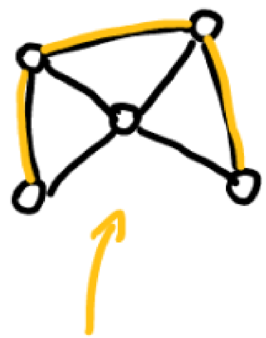


P_4- free



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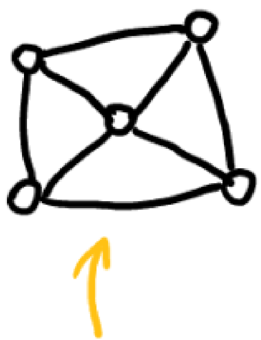

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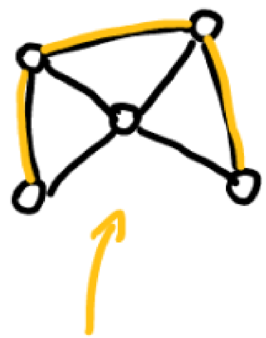

not P_4-free

hereditary graph class:

G is F-free $\Rightarrow$ $G-v$ is F-free

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- For which pairs (F, H) we can solve HOM(H) and LHOM(H) faster if our instances are F-free?

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since H fixed, we check this in constant time $\Rightarrow$ HOM(H) trivial for (P_3, H) (for any H)

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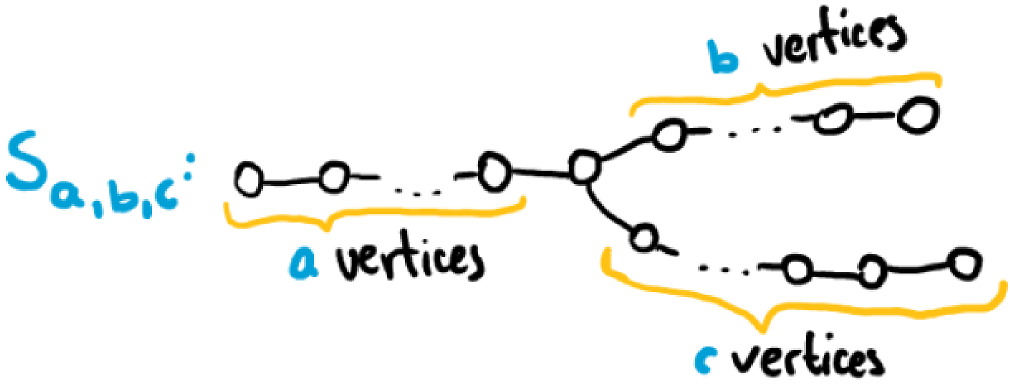
e.g. (non-trivial)

HOM(K_3) and LHOM(K_3) polynomial-time solvable in P_7-free graphs [4]

WHAT WE KNEW



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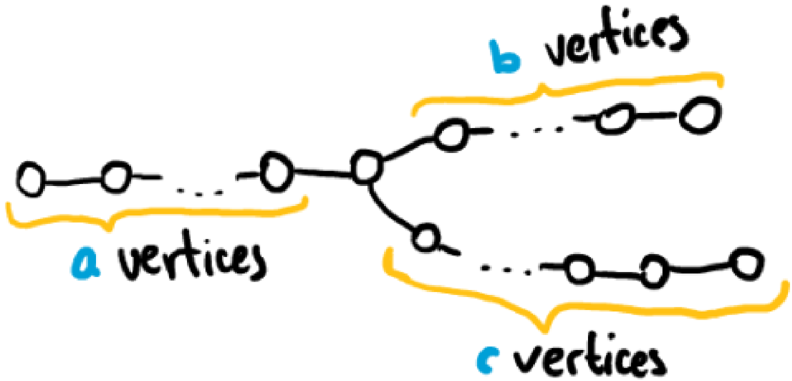
$S_{a,b,c}$:



H non-bi-arc, $F \neq P_t$, $F \neq S_{a,b,c}$ [5]

$\rightarrow$ LHom(H) ETH-hard even in F-free graphs

WHAT WE KNEW

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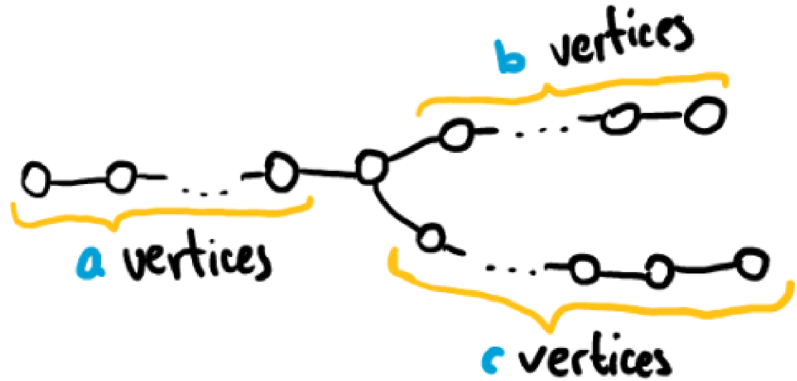

Remaining cases:

- complexity of HOM(H) in F-free graphs*

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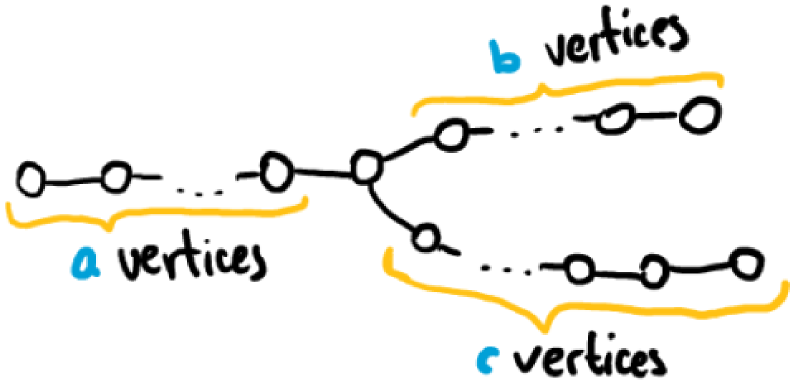

Remaining cases:

- complexity of $\text{HOM}(H)$ in F-free graphs*
- complexity of $\text{LHOM}(H)$ in P_t-free graphs or in $S_{a,b,c}$-free graphs

  algorithm for $\text{LHOM}(H)$ would work for $\text{HOM}(H)$

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induced subgraph

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- so we can assume $K_3 \not\leq H$

but H still can have e.g., 

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$\Rightarrow$ we solve $\textsc{LHom}(H)$ in subexponential-time

if $K_3 \leq G$ then no-instance
(no $\triangle \to H$)

if $K_3 \not\leq G$
we use the algorithm

$S_{a,b,c}$-graphs:

- if H contains loops $u,v,w \in V(H)$ and no $x,y \in \{u,v,w\}$ s.t. $N(x) \subseteq N(y)$ we can show there exist a',b',c' s.t. $LHom(H)$ ETH-hard in $S_{a,b,c}$-free graphs for $a \geq a'$, $b \geq b'$ and $c \geq c'$.

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open case: mixed graphs H (with loops just on some vertices)

[1] Hell, Nešetřil, JCTB 1990

[2] Cygan et al, SODA 2016

[3] Feder, Hell, Huang, JGT 2003

[4] Bonomo et al, Combinatorica 2018

[5] Piecyk, Rzążewski (pers. communication)

[6] Chudnovsky et al, SODA 2020