On The Spectral Reconstruction Problem For Digraphs

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Introduction

In 1957, Paul Kelly wrote his doctoral thesis under the suppervision of Stanislaw Ulam. His thesis proved that the *Reconstruction Construction* is true for trees. Three years later, Ulam published the statement of the *Reconstruction Conjecture*.

The Reconstruction Conjecture

Let G and H be graphs with $V(G) = \{v_1, \ldots, v_n\}$ and $V(H) = \{u_1, \ldots, u_n\}$ for $n >= 3$. If $G - v_i \equiv H - u_i$ for $i = 1, \ldots, n$ then $G \equiv H$.

There are many classes which are known to be reconstructible:

- Regular graphs
- Trees
- Disconnected graphs
- Maximal planar graphs

Variants of The Reconstruction Conjecture (RC I)

Let $\mathcal{I}$ be a graph invariant. That is, if G and H are two isomorphic graphs, then $\mathcal{I}(G) = \mathcal{I}(H)$. We say that $\mathcal{I}$ is reconstructible if it is uniquely determined by the deck.

Theorem: [Tutte, 1979]

The characteristic polynomial and the chromatic number are reconstructible.

Many other properties are known to be recostructible: The order of the graph, the number of edges, the degree sequence, Tutte polynomial, Planarity...

Variants of The Reconstruction Conjecture (RC II)

A natural question to ask is whether a graph polynomial is reconstrictible from its polynomial deck, that is, from the multiset of the polynomials of the vertex-deleted subgraphs.

Question: Cvetkovic(1973)

Is the characteristic polynomial reconstructible from the characteristic polynomial of the deck.

This question is still open. Hagos [Hagos, 2000] proved that the characteristic polynomial is reconstructible from of a graph is reconstructible from its polynomial deck together with the polynomial deck of its complement.

The Idiosyncratic Polynomial

Definition 1

Let G be a graph and let A be its adjacency matrix. The *Idiosyncratic Polynomial* of G is the characteristic polynomial of $A + y(J - A - I)$.

Theorem 2 ([Johnson and Newman, 1980])

Two graphs have the same idiosyncratic polynomial if only if they are cospectral, and their complements are also cospectral.

This theorem, combined with Hagos'result, shows that the idiosyncratic polynomial of a graph is recontructible from its idiosyncratic polynomial deck.

- Stockmeyer [Stockmeyer, 1977] contructed two non-isomorphic tournaments on 2^n vertices which have the same deck $\rightarrow$ RC is false.
- Dumont(1979) checked that for $n \leq 6$, the difference (in absolute value) between the determinants of these tournaments is 1.

By inspecting the number of hamiltonian cycles in Stockmeyer's tournaments, we get the following result.

Proposition 1

For $n \geq 3$, the determinants of Stockmeyer tournaments do not have the same parity.

Corollary 3

RC I is false for digraphs.

Fraïssé's Problem

Fraïssé [Fraïssé, 1970] considered a strengthening of the reconstruction conjecture for the class of relations which contains graphs and digraphs. For digraphs, Fraïssé's problem can be stated as follows.

Question 1

Let G and H be two digraphs with the same vertex set V and assume that for every proper subset W of V, the subdigraphs $G[W]$ and $H[W]$, induced by W, are isomorphic. Is it true that G and H are isomorphic?

The answer is positive when $|V| \geq 7$ [Lopez, 1978]. It follows that if $G[W]$ and $H[W]$ are isomorphic for every subset W of size at most 6, then G and H are isomorphic.

Variant of Fraïssé's Question

Motivated by Lopez'theorem, we can ask the following question:

Question 2

Let $\mathcal{I}$ be a digraph invariant polynomial and let G be a digraph. Is the polynomial $\mathcal{I}(G)$ reconstructible from the collection $\{\mathcal{I}(H) : H \in \mathcal{H}\}$, where $\mathcal{H}$ is the set of proper induced subdigraphs of G?

We address this question in the case of the idiosyncratic polynomial for digraphs.

Definition 4

Let G be a digraph and let A be its adjacency matrix. The *idiosyncratic polynomial* of G is the characteristic polynomial of $A + y(J - A - I) + zA^t$.

- Because digraphs are not a symmetric relation, the term zA^t is added to gives information about the converse.
- If two digraphs have the same idiosyncratic polynomial then they have the same characteristic polynomial. Moreover, their complements and their converses have also the same characteristic polynomial.

Main Result

- Question 2 is not true for arbitrary digraphs: the counterexamples contains one of two particular digraphs called flags [Boussaïri et al., 2004].
- For flag-free digraph, we have the following result:

Theorem 5

Let G and H be two flag-free digraphs with the same vertex set V of size at least 5. If for every 3-subset W of V, the induced subdigraphs $G[W]$ and $H[W]$ have the same idiosyncratic polynomial, then G and H have the same idiosyncratic polynomial.

Sketch of the proof

Two digraphs D_1 and D_2 with the same vertex set V are k-hemimorphic if for every k-subset W of V, $D_1[W]$ is isomorphic to $D_2[W]$ or its converse. The proof proceeds as follows:

- It is easy to verify that digraphs on 3-vertices are determined by their idiosyncratic polynomial, hence the conditions of the theorem are verified if G and H are 3-hemimorphic.
- Prove that two 3-hemimorphic digraphs are 2-hemimorphic.
- $\{2, 3\}$-hemimorphic digraphs are related by special relation (interval reverals)[Boussaïri et al., 2004].
- Show that the idiosyncratic polynomial is invariant under interval reversals.

Consequences of The Main Theorem

- Two tournaments satisfy the assumption of Theorem 5 if and only if they have the same 3-cycles.
- Using [Gregory et al., 1993, Lemma 5.2], it easy to check that two cospectral tournaments have the same idiosyncratic polynomial.

Corollary 6

Two tournaments with the same 3-cycles have the same characteristic polynomial.

Posets and Comparability Graphs

- Posets are an important class of digraphs for which the Reconstruction Problem is still open.
- Ille and Rampon [Ille and Rampon, 1998] proved that a poset is reconstructible by its deck together with its comparability graph.

A parameter of a poset is said to be a comparability invariant if all posets with a given comparability graph have the same value of that parameter (e.g. dimention, number of transitive extension). From Theorem 5 we have

Corollary 7

The idiosyncratic polynomial is a comparability invariant.

Questions

Let G be a comparability graph and let I be the common idiosyncratic polynomial of its transitive orientations. If the idiosyncratic polynomial of an orientation of G is equal to I then it is transitive.
This is not true if we consider the characteristic polynomial of the skew-adjacency matrix.

Question 3

Let G be a comparability graph and let ζ be the common skew-characteristic polynomial of its transitive orientations. What is the relation between the orientations of G whose skew-characteristic polynomial is ζ.

- They are switching equivalent if G is a complete graph [Deng et al., 2018] or bipartite graph [Anuradha et al., 2013].
- If G is an odd-cycle graph, then all its orientations have the same skew-characteristic polynomial [Cavers et al., 2012].

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