There are no first-order sentences with quantifier depth 4 and an infinite spectrum

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8th Polish Combinatorial Conference. September 14-18, 2020

Outline

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Erdős–Rényi random graph model

Definition

Erdős–Rényi random graph model is a probabilistic space

$$G(N,p) = (\Omega_N, \mathcal{F}_N, \mathsf{P}_{N,p}),$$

where $N \in \mathbb{N}$, $0 \leqslant p \leqslant 1$,

$$\Omega_N = \{\mathcal{G} = (\mathcal{V}_N, \mathcal{E})\} -$$

set of all graphs with $\mathcal{V}_N = \{1, 2, \ldots, N\}$,

$$\mathcal{F}_N = 2^{\Omega_N}, \quad \mathsf{P}_{N,p}(\mathcal{G}) = p^{e(\mathcal{G})}(1-p)^{C_N^2 - e(\mathcal{G})}.$$

First-order graph properties

Definition

First-order graph property is a property defined by the first order formula with the folowing symbols:
— predicate symbols: $=$, $\sim$;
— logical symbols: $\rightarrow$, $\wedge$, $\vee$, $\neg$, ...;
— variables: x, y, ...;
— quantifiers: $\exists$, $\forall$.

Definition

Quantifier depth of a first-order property Q is a minimal quantifier depth of a first-order formula that expresses Q.

Zero-One Law

Definition

For $p = p(N)$ the Zero-One Law holds, if for each first-order property L $\mathsf{P}_{N,p}(L)$ tends to either 0 or 1.

Consider probabilities $p(N) = N^{-\alpha}$.

Theorem (J. Spencer, S. Shelah, 1988)

Let $p(N) = N^{-\alpha}$.
— Let α be a positive irrational. Then for $p(N)$ a Zero-one law holds.
— Let α be a positive rational. If $\alpha > 2$ or $\alpha \in \left(1 + \frac{1}{l+1}, 1 + \frac{1}{l}\right)$ for some $l \in \mathbb{N}$, then for $p(N)$ the Zero-One Law holds. In all other cases the Zero-One Law for $p(N)$ does not hold.

Zero-One k-Law

Definition

For $p = p(N)$ the Zero-One k-Law holds, if for each first-order property L with quantifier depth $\leqslant k$ $\mathsf{P}_{N,p}(L)$ tends to either 0 or 1.

Theorem (M. Zhukovskii, 2012)

Let $p(N) = N^{-\alpha}$. If $\alpha \in \left(0, \frac{1}{k-2}\right)$, then for $p(N)$ a Zero-One k-Law holds.
If $\alpha = \frac{1}{k-2}$, then for $p(N)$ the Zero-One k-Law does not hold.

Fisrt-order spectra

Definition

We call a k-spectrum a set of all $\alpha \in (0,1)$ s.t. for $p(N) = N^{-\alpha}$ the Zero-One k-Law does not hold.

Consider a FO property Q.

Definition

Spectrum of Q is a set of all $\alpha \in (0,1)$ s.t. $\mathsf{P}_{N,p}(Q)$ does not tend to either 0 or 1 for $p(N) = N^{-\alpha}$.

A k-spectrum is clearly a union of all spectra of FO formulas Q with quantifier depth $\leqslant k$.

Example: simple formula spectrum

- Consider property $Q = \{G \text{ has an induced } C_4\} \triangle \{G \text{ has an induced } K_4\}$.
- By the Bollobás theorem, $p_1 = \frac{1}{N}$ and $p_2 = N^{-\frac{2}{3}}$ are threshold probabilities of respective parts
- Hence, the spectrum equals $\left\{\frac{2}{3}, 1\right\}$ (well, not really)

Infinity of spectrum

Theorem (J. Spencer, 1990)

There exists a FO property of depth 14 with an infinite spectrum.

Theorem (M. Zhukovskii, 2016)

There exists a FO property of depth 5 with an infinite spectrum. The 5-spectrum is finite.

Theorem (M. Zhukovskii, A. Matushkin, 2017)

The only possible limiting points of 4-spectrum are $1/2$ and $3/5$.

4-spectrum finiteness

Theorem (A. Matushkin, M. Zhukovskii, Y.Y.)

Points $\frac{1}{2}$ and $\frac{3}{5}$ cannot be limiting in 4-spectrum. Therefore, 4-spectrum is finite.

Outline

2. Required theorems and constructions

Ehrenfeucht game

We now give a criterion of validity of the zero-one k-law.

Theorem (A. Ehrenfeucht, 1960)

Random graph $G(N, p)$ obeys the zero-one k-law if and only if

$$\lim_{N,M\to\infty} \mathsf{P}_{N,M,p}\Big(\{(A, B) : \text{Duplicator has a winning strategy in gameEHR}(A, B, k)\}\Big) = 1,$$

где $\mathsf{P}_{N,M,p}$ is a product of measures $\mathsf{P}_{N,p}$ и $\mathsf{P}_{M,p}$.

Threshold probabilities of properties "contain a copy of the subgraph"

Let

$$\rho(G) = \frac{e(G)}{v(G)};$$

$$\rho^{\max}(G) = \max_{H \subseteq G} \rho(H).$$

Theorem (Ruciński A., Vince A, 1985)

Let $p_0(N) = N^{-1/\rho^{\max}(G)}$. If $p = o(p_0)$ then $G(N,p)$ a.a.s. does not contain a copy of G. If $p_0 = o(p)$ then, on the contraty, $G(N,p)$ a.a.s. does not contain a copy of G.

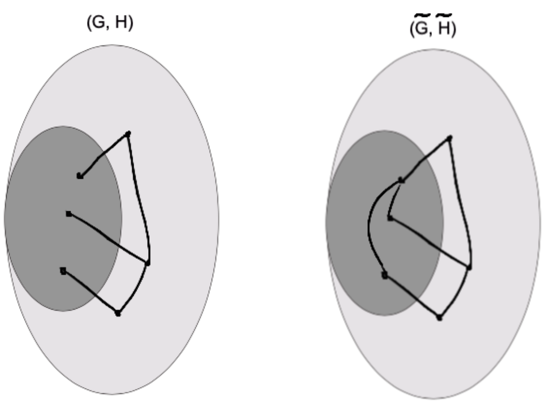
Extensions

Definition

Let (G, H) and $(\tilde{G}, \tilde{H})$, $G \subset H$, $\tilde{G} \subset \tilde{H}$ be two pairs of graphs. Let

$$V(G) = \{x_1, \ldots, x_m\}, V(H) = \{x_1, \ldots, x_l\},$$
$$V(\tilde{G}) = \{\tilde{x}_1, \ldots, \tilde{x}_m\}, V(\tilde{H}) = \{\tilde{x}_1, \ldots, \tilde{x}_l\}.$$

If $(x_i, x_j) \in E(G) \setminus E(H) \Rightarrow (\tilde{x}_i, \tilde{x}_j) \in E(\tilde{G}) \setminus E(\tilde{H})$, then $\tilde{G}$ is called a (G, H)-extension of $\tilde{H}$.



Extensions

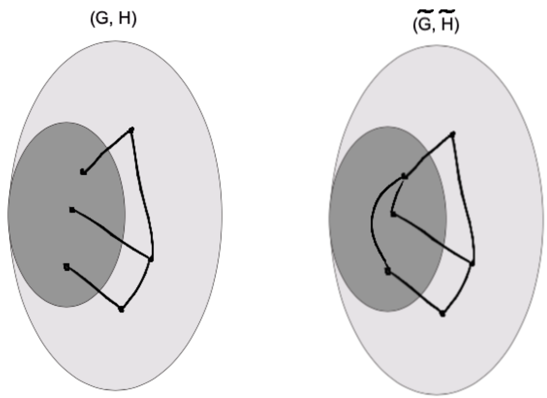
Fix $\alpha > 0$. Let

$$v(G,H) = v(G) - v(H), e(G,H) = e(G) - e(H),$$
$$f_\alpha(G,H) = v(G,H) - \alpha e(G,H).$$

α-safe extensions

$$f_\alpha(G,H) = v(G,H) - \alpha e(G,H).$$

Pair (G,H) is called α-safe, if $\forall S \quad (H \subset S \subseteq G \to f_\alpha(S,H) > 0)$.



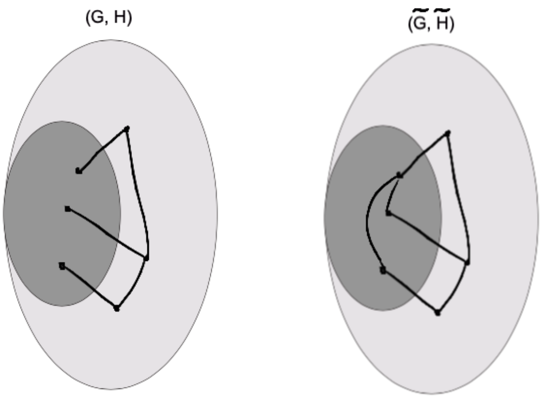
$\frac{1}{2}$-safe

If the pair (G,H) is α-safe then a.a.s. there is (there are many) a (G,H)-extension of each subgraph $\tilde{H}$ in the random graph $G(N,p)$.

α-rigid extensions

$$f_\alpha(G, H) = v(G, H) - \alpha e(G, H).$$

Pair (G, H) is called α-rigid, if $\forall S \quad (H \subseteq S \subset G \to f_\alpha(G, S) < 0)$.

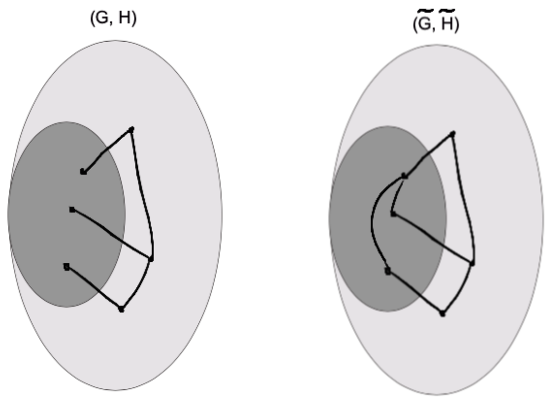


$\frac{1}{3}$-rigid

α-neutral extensions

$$f_\alpha(G, H) = v(G, H) - \alpha e(G, H).$$

Pair (G, H) is called α-neutral, if $\forall S \quad (H \subset S \subset G \to f_\alpha(S, H) > 0)$ и $f_\alpha(G, H) = 0$

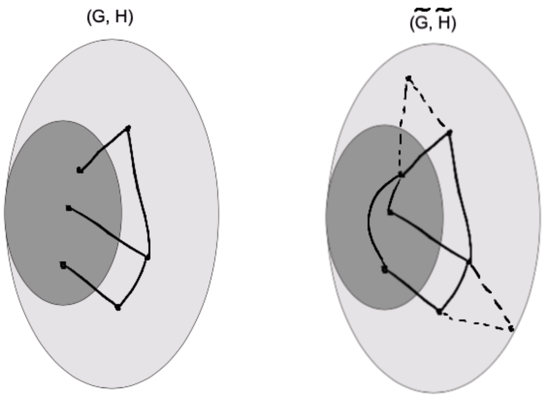


$\frac{3}{5}$-neutral

Maximal extensions

Question:
Can there be any α-neutral or α-rigid extensions in a random graph?

Answer:
Yes, there can be. But we can choose new vertices so that there are none.



$$\alpha \in \left(\frac{1}{2}, \frac{1}{2} + \varepsilon\right)$$

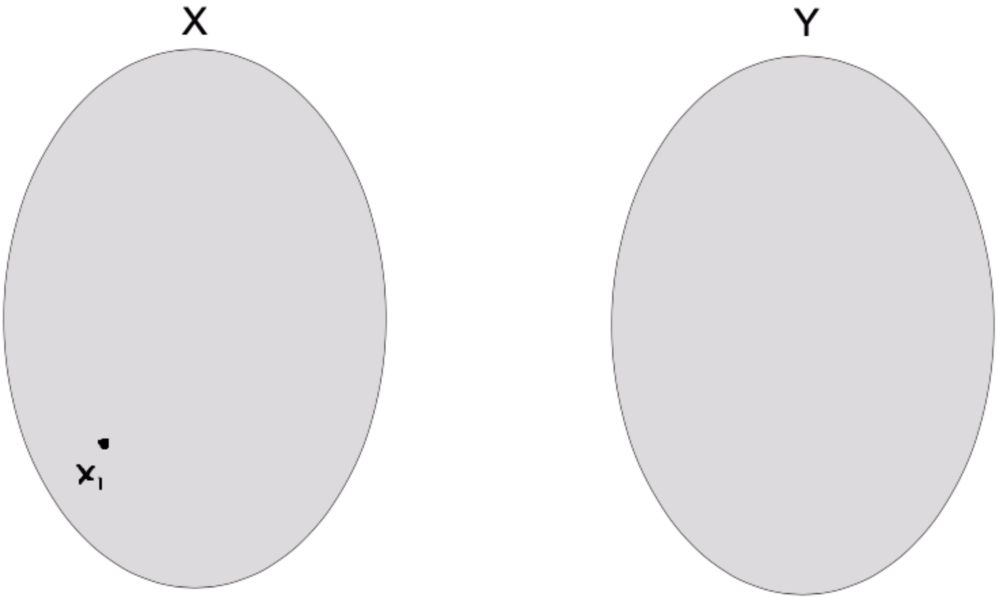
We can choose a (G, H)-extension such that there are no outer vertices of degree $\geqslant 2$

Outline

3. How to play the Ehrenfeucht game

How to play the Ehrenfeucht game

$$\alpha \in (\tfrac{1}{2}, \tfrac{1}{2} + \varepsilon)$$



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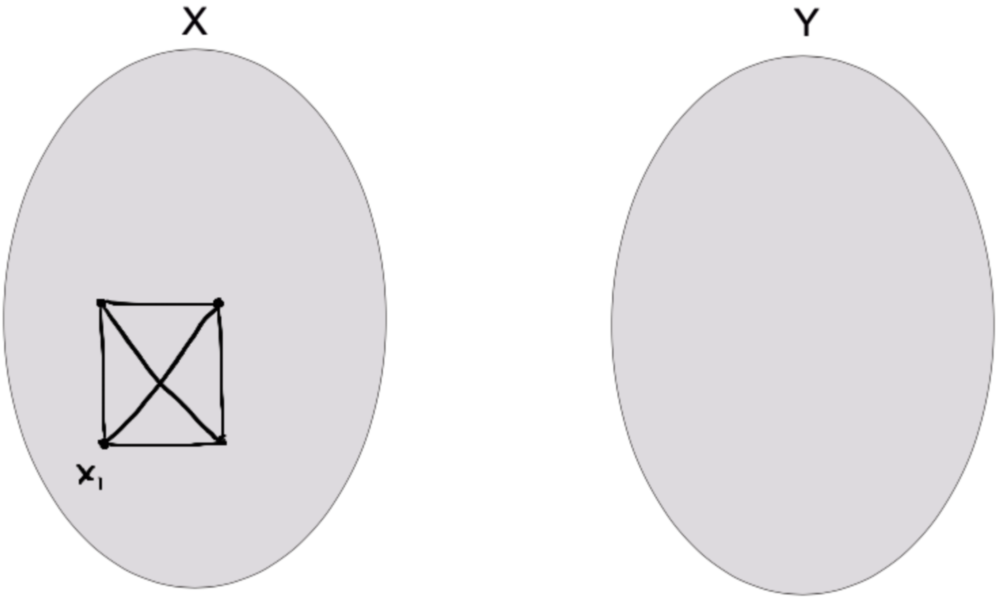

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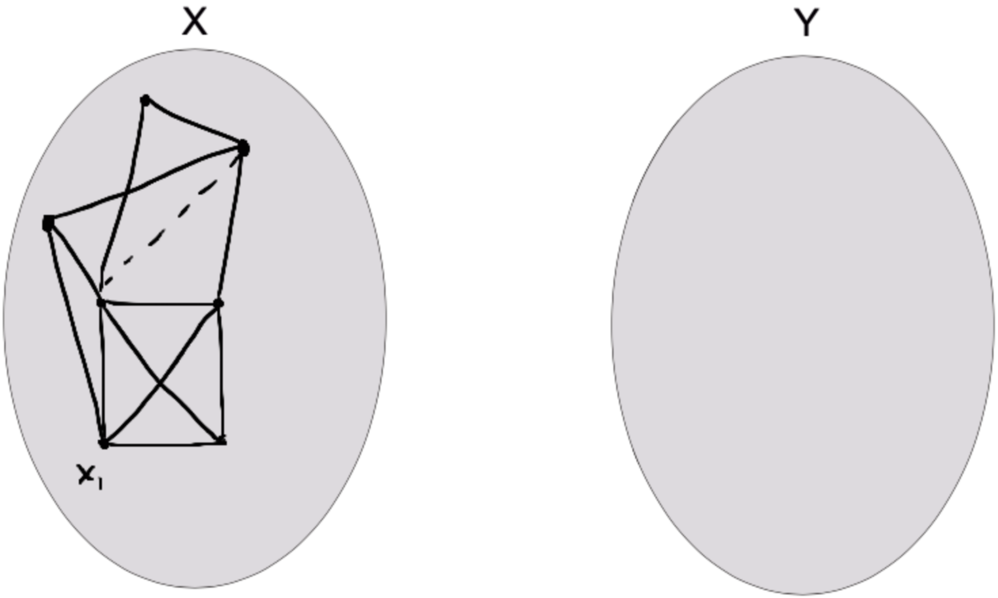

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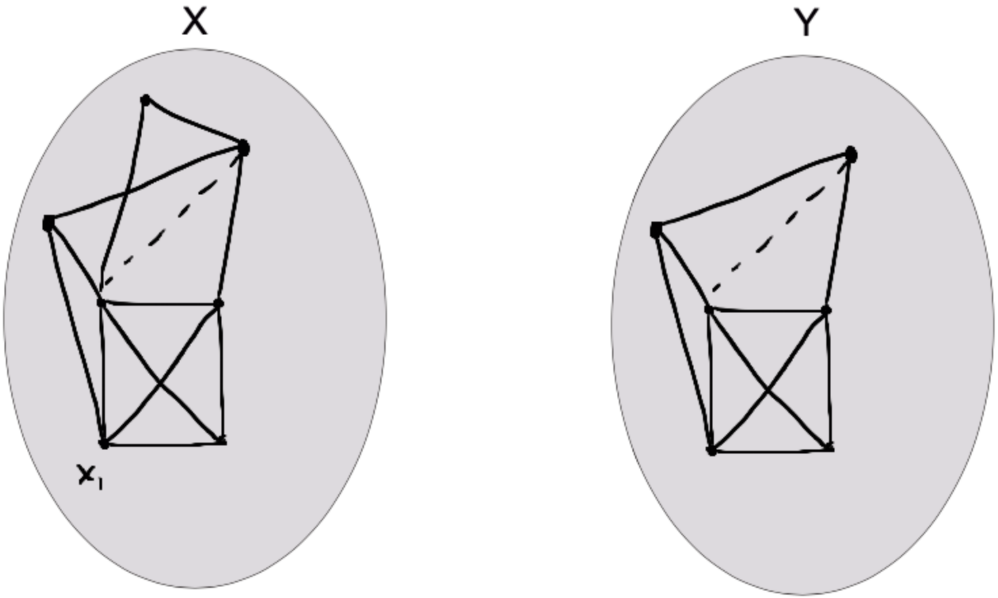

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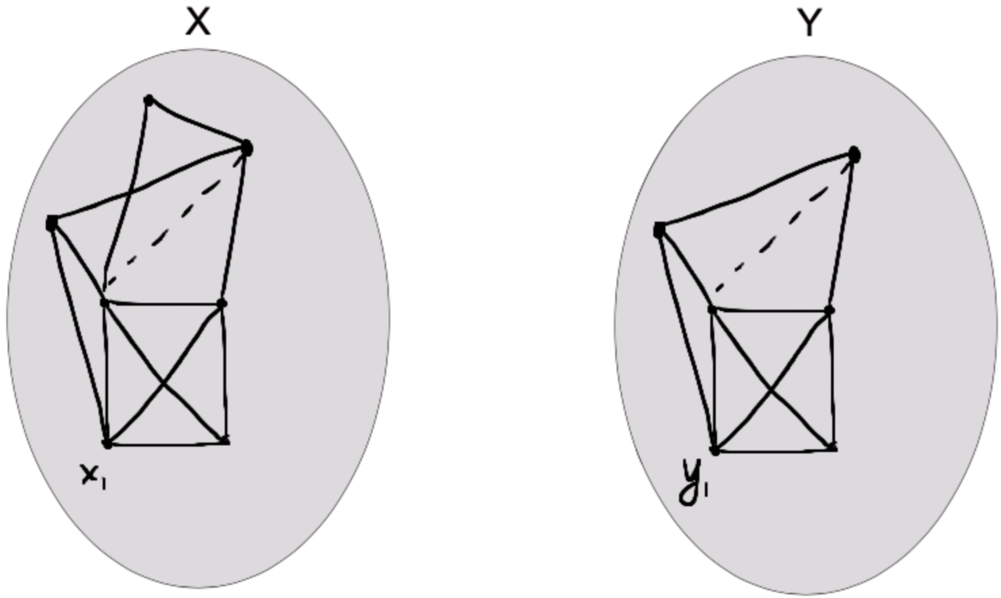

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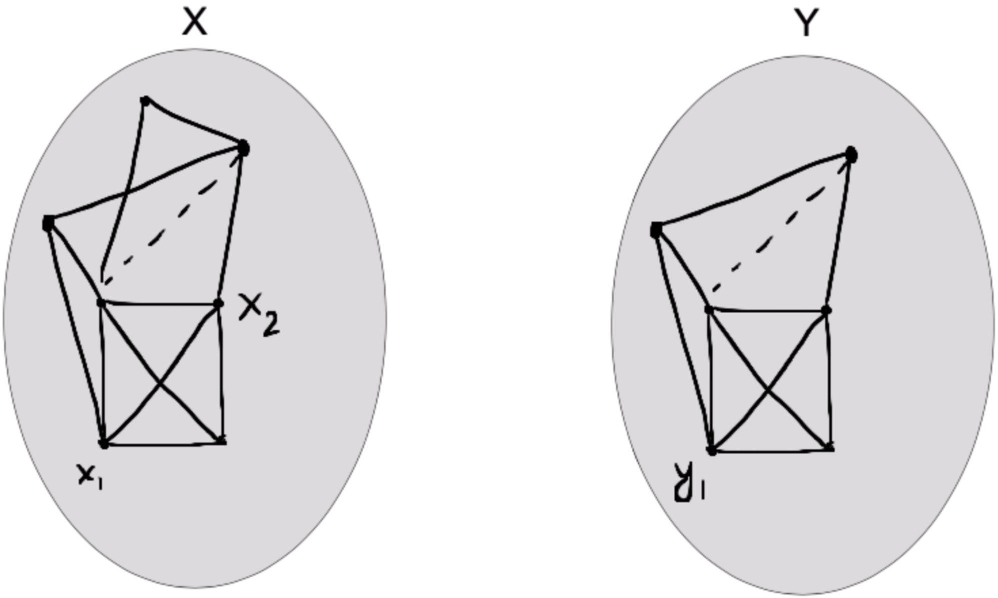

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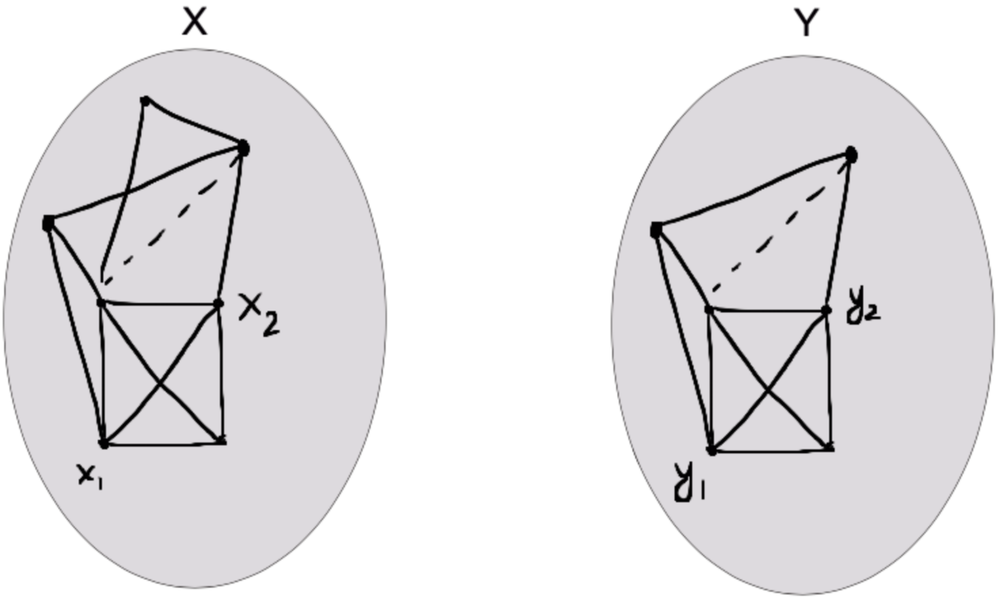

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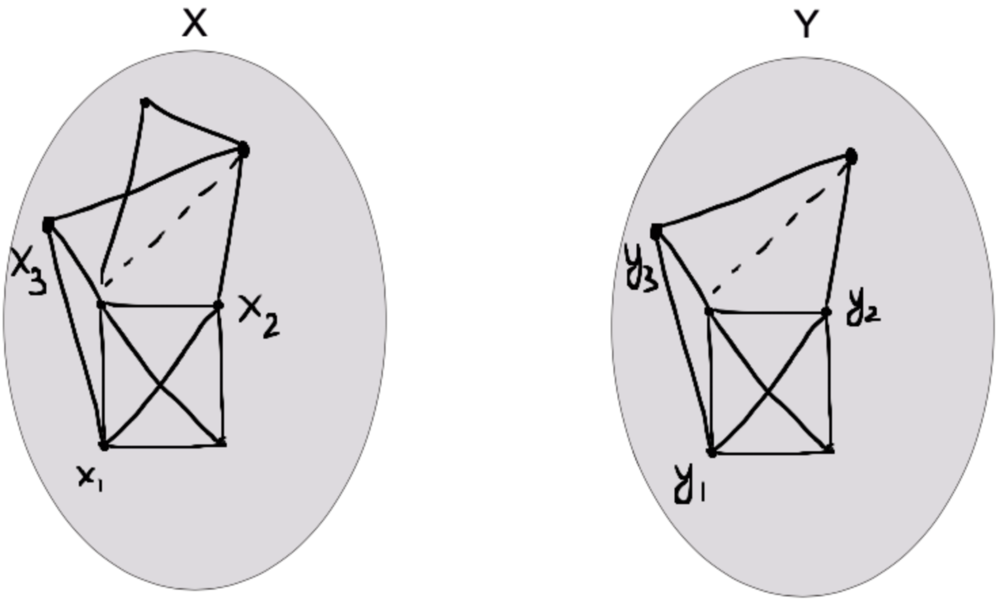

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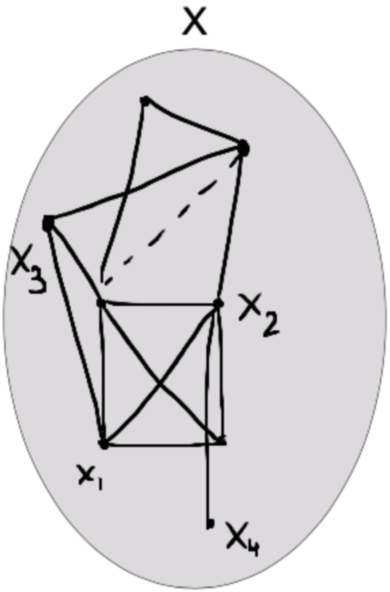

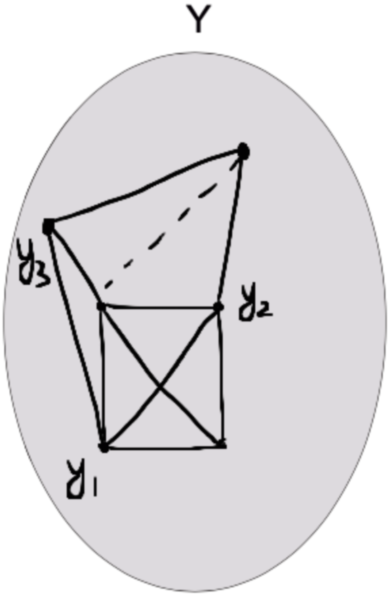

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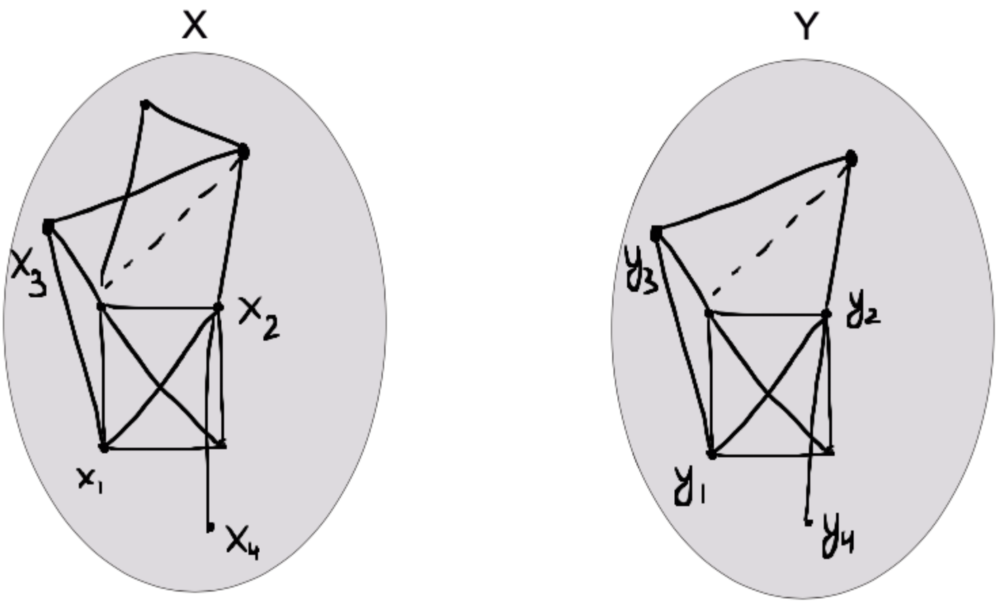


Outline

4. Proof sketch for $\frac{1}{2}$

Proof sketch

- We prove that there exists $\varepsilon > 0$ s.t. for each $\alpha \in (\frac{1}{2}, \frac{1}{2} + \varepsilon)$ a.a.s. in measure P_α Duplitator wins.
- We give a strategy for Duplicator to win a.a.s. in $\mathrm{EHR}(X, Y, 4)$.
- Let $x_1 \in V(X)$ be the Spoiler's first step, let U be the maximal dense subgraph of X containing x_1 that "spoils" Duplicator's play.
- Since $U \subseteq X$, we have a.a.s. $\rho^{\max}(U) < \frac{1}{\alpha}$.
- There is an "almost copy" W of graph U in Y, which is maximal.
- Duplicator chooses $y_1 \in V(Y)$, which is the isomorphic image of x_1.
- Similarities between U и W provide for the win of Duplicator.

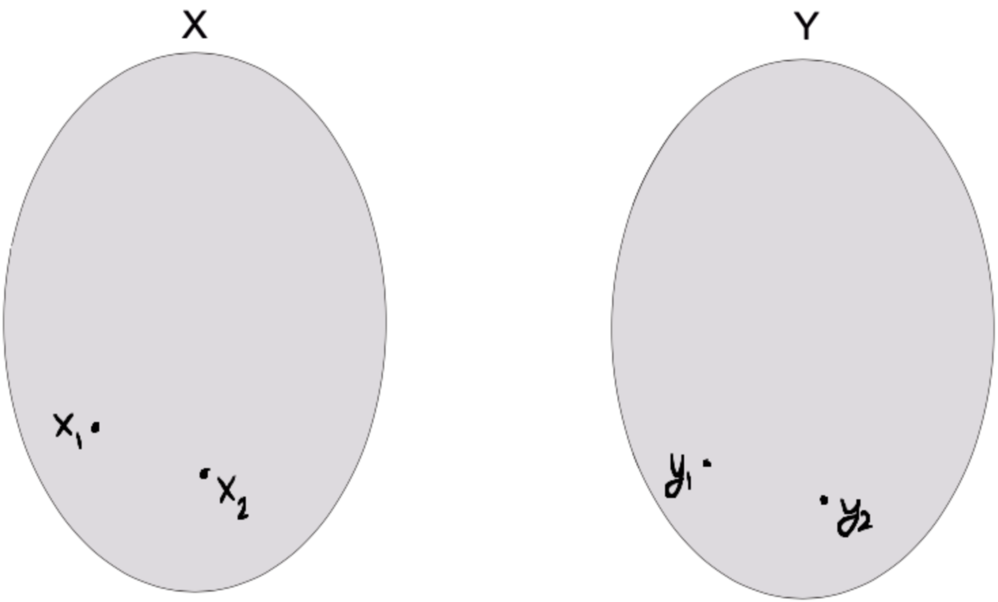
Outline

How to win for Spoiler

- Consider $\alpha = \frac{1}{2}$
- We prove that α is a limiting point in 5-spectrum
- We give a strategy for Spoiler to win in $\mathrm{EHR}(X, Y, 5)$
- This proves the existence of the formula of depth 5 with an infinite spectrum.

How to win for Spoiler

$$\alpha \in (\tfrac{1}{2}, \tfrac{1}{2} + \varepsilon)$$



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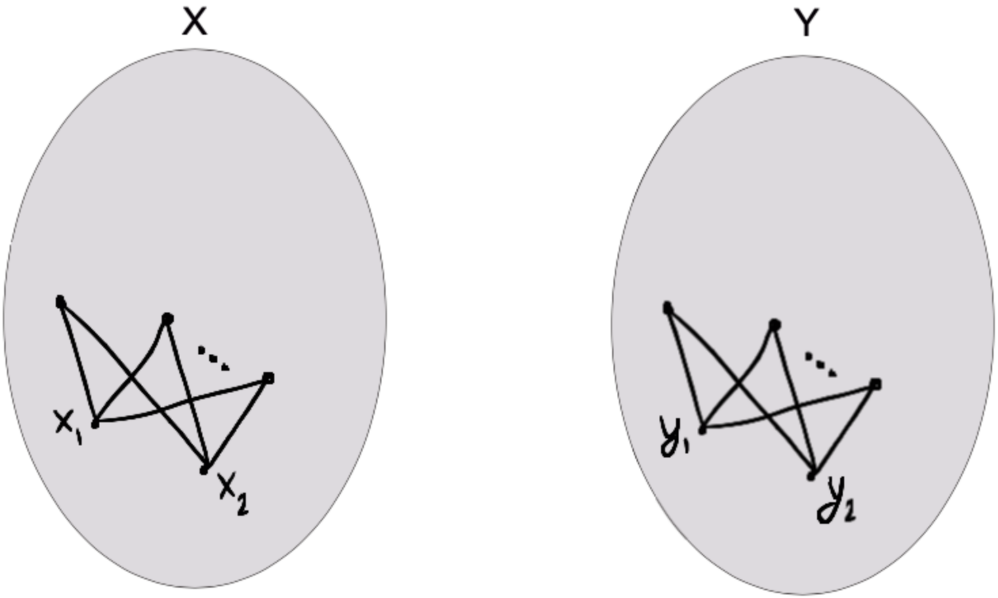

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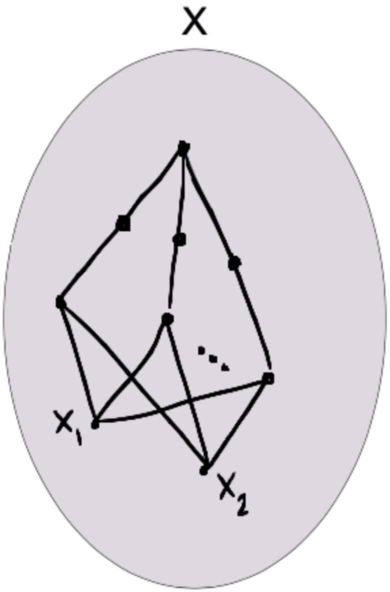

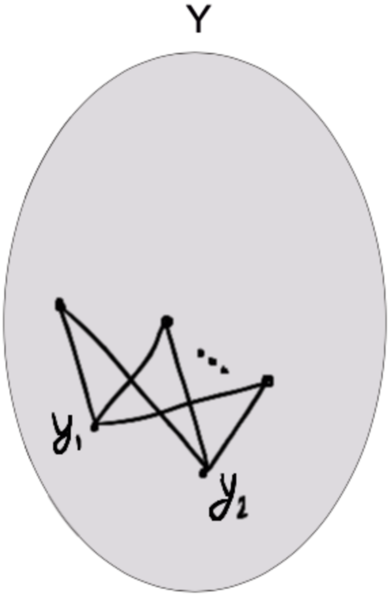

Outline

6 Future research

Future research

- Explain the structure of k-spectrum.
- Study the limiting points of k-spectrum for arbitrary k
- At least understand whether $\frac{1}{k-2}$ is limiting

Thank you for your attention!