Ryser's Conjecture for t-intersecting hypergraphs

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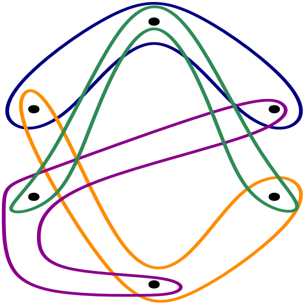

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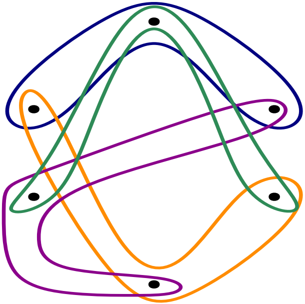

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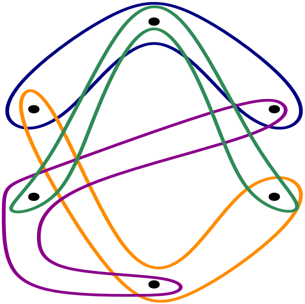

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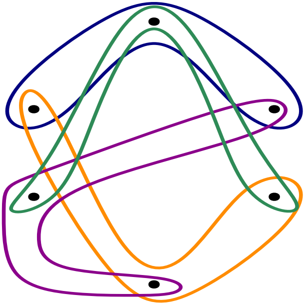

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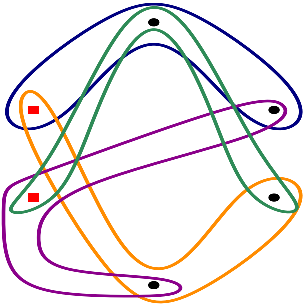

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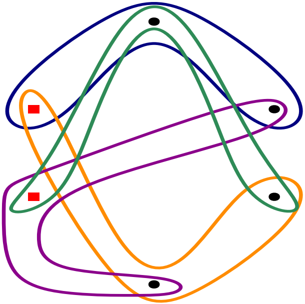

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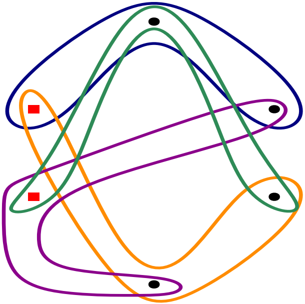

- $C \subseteq V$ a cover if
  $C \cap e \neq \emptyset, \forall e \in E$
- Cover number:
  $\tau(\mathcal{H}) := \min_{\{C \subseteq V : C \text{ a cover}\}} |C|$

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For *all* hypergraphs $\mathcal{H}$:

$$\nu(\mathcal{H}) \leq \tau(\mathcal{H}) \leq r\nu(\mathcal{H})$$

r-PARTITE HYPERGRAPHS

An r-uniform $\mathcal{H} = (V, E)$ is r-partite if there exists a partition $V = V_1 \sqcup \ldots \sqcup V_r$ such that

$$|e \cap V_i| = 1,$$

for all $i \in [r]$ and $e \in E$.

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Example

Bipartite graphs

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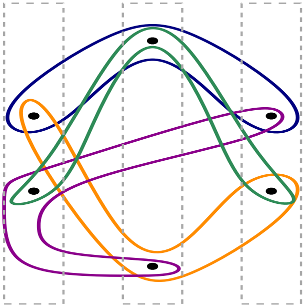

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Conjecture (Ryser's conjecture)

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- True for $r \leq 5$ if $\nu(\mathcal{H}) = 1$ (Tuza 1979)$\longrightarrow \tau(\mathcal{H}) \leq r-1$.

t-INTERSECTING HYPERGRAPHS

An r-uniform hypergraph $\mathcal{H} = (V, E)$ is t-intersecting if

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$$\text{Ryser}(r,t) := \max \left\{ \tau(\mathcal{H}) : \mathcal{H} \text{ an } \begin{array}{c} r\text{-uniform,} \\ r\text{-partite,} \\ t\text{-intersecting} \end{array} \text{ hypergraph} \right\}.$$

Bounding Ryser(r, t)

Conjecture (Bustamante-Stein 2018, Király-Tóthmérész 2017)

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- $\mathrm{Ryser}(r,t) \geq \lfloor r/t \rfloor - 1$

Our results

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Theorem (Bishnoi-Das-M.-Szabó 2020+)

For all $1 \leq t \leq r-1$,

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For all $1 \le t \le r-1$,

$$\mathrm{Ryser}(r,t) \le \begin{cases} \left\lfloor \frac{r-t}{2} \right\rfloor + 1 & \text{if} \quad t \ge (r+1)/3, \\ 2r - 5t + 2 & \text{if} \quad 7r/26 \lesssim t \lesssim (r+1)/3, \\ \frac{9r-14t}{8} + 2 & \text{if} \quad r/5 \lesssim t \lesssim 7r/26, \\ \frac{15r-44t}{8} + 3 & \text{if} \quad 9r/52 \lesssim t \lesssim r/5. \end{cases}$$

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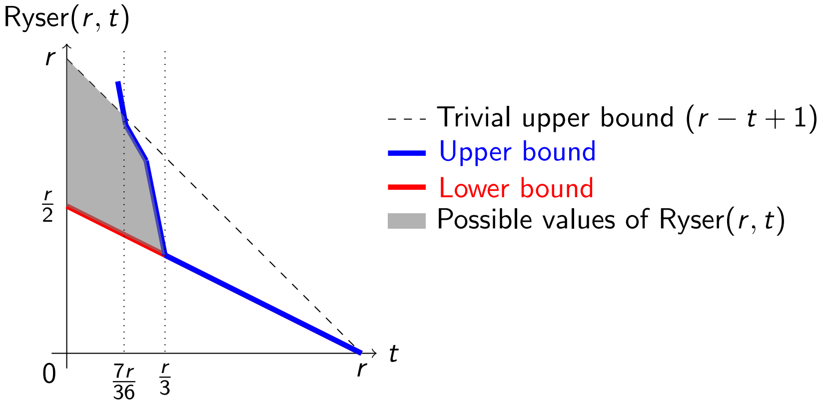

Figure: The value of Ryser(r,t) for large r.

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Conjecture (Bishnoi-Das-M.-Szabó 2020+)

For all $2 \leq t \leq r$,

$$\mathrm{Ryser}(r,t) = \left\lfloor \frac{r-t}{2} \right\rfloor + 1.$$

The construction

AIM: Construct a $\mathcal{H}$ which

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Let

$$\binom{[r]}{r-\ell} =: \{S_1, \ldots, S_m\},$$

(so that $m = \binom{r}{r-\ell}$).

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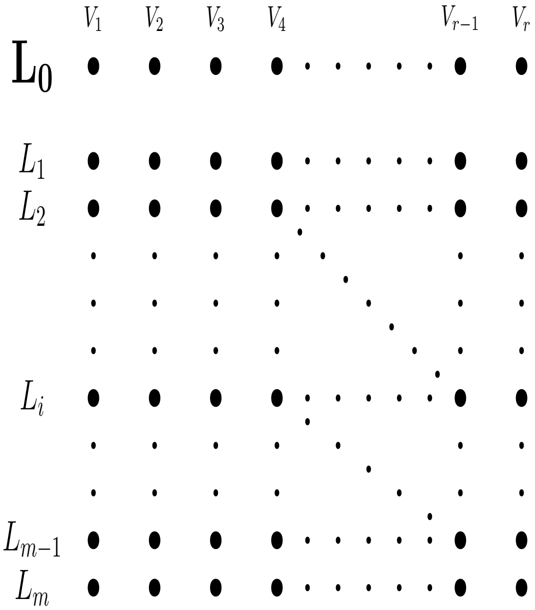

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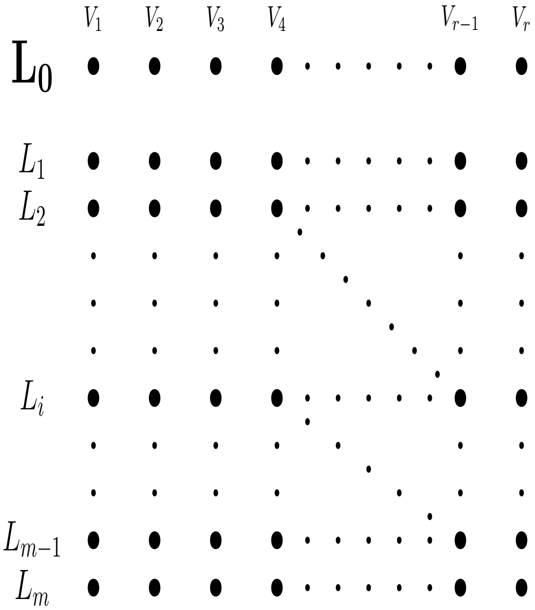

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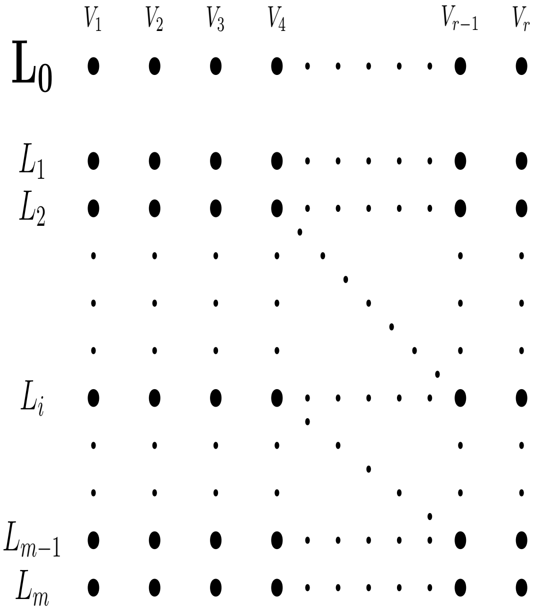

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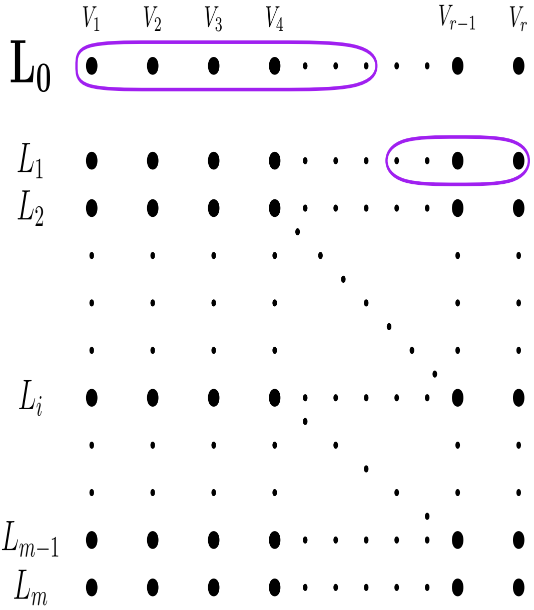

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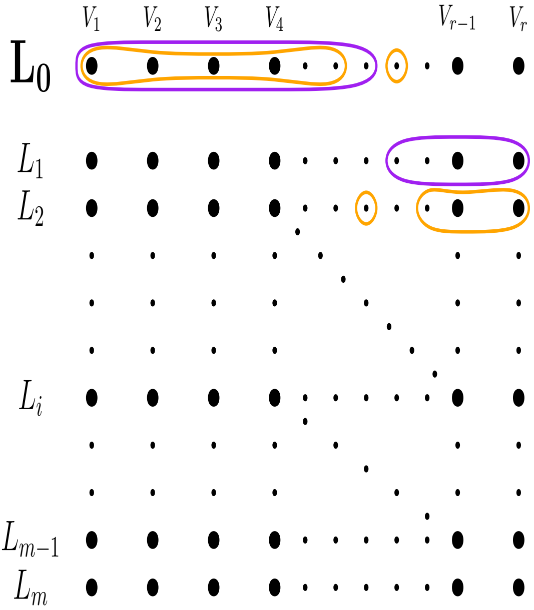

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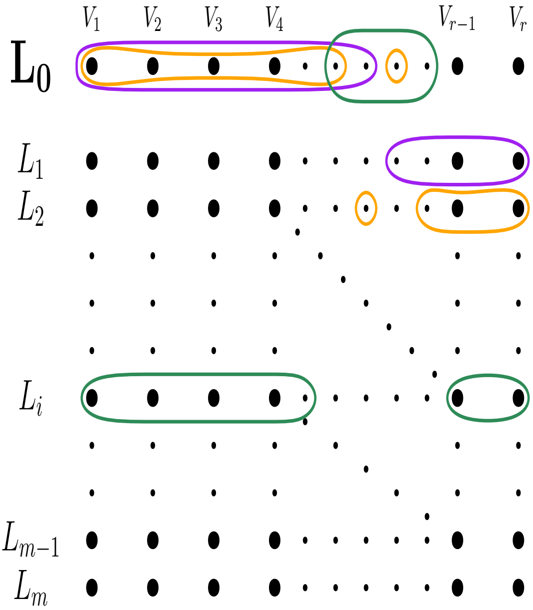

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   - $e_i^* \cap C = \emptyset$, a contradiction.

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We show: If $(r+1)/3 \le t \le r-1$, then $\mathrm{Ryser}(r,t) \le s := \left\lfloor \frac{r-t}{2} \right\rfloor + 1$.

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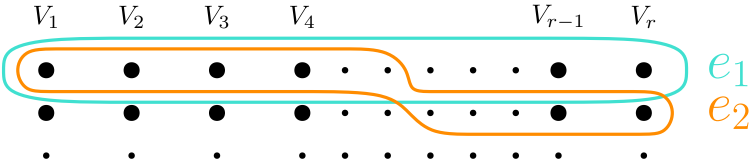

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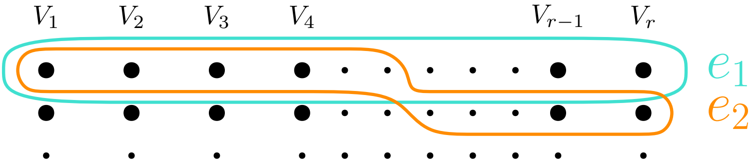

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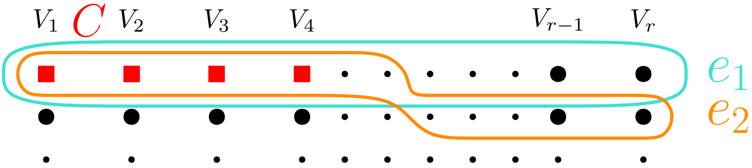

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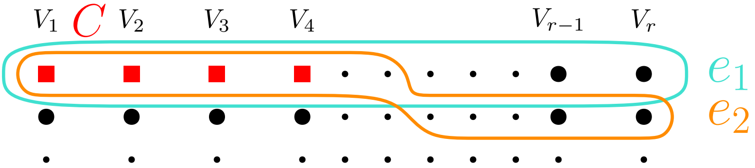
- If C not a cover, there exists $e_3 \in E(\mathcal{H})$ with $e_3 \cap C = \emptyset$



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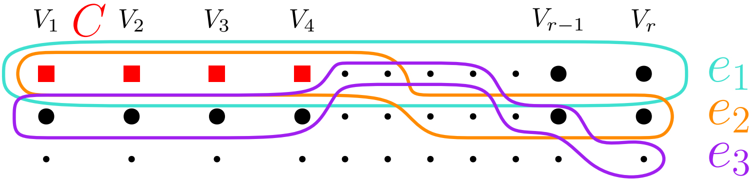

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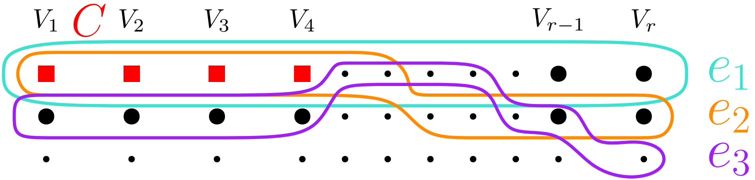

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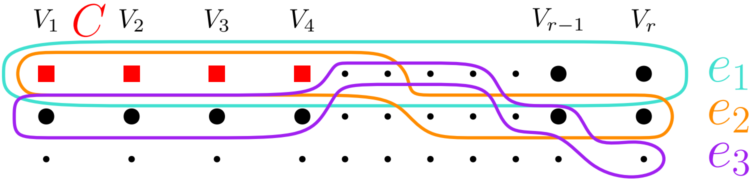

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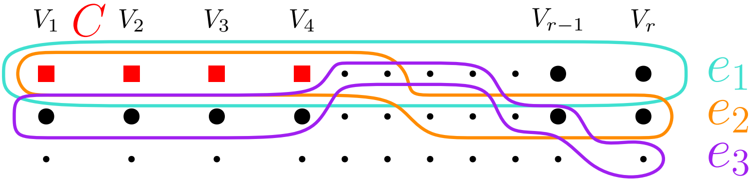

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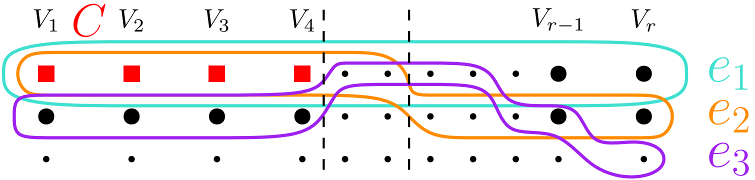

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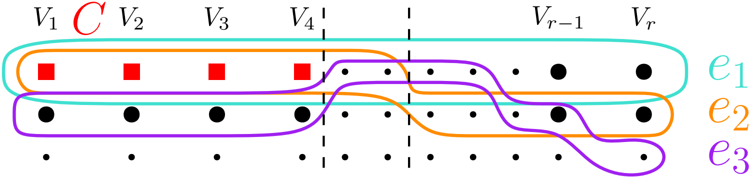

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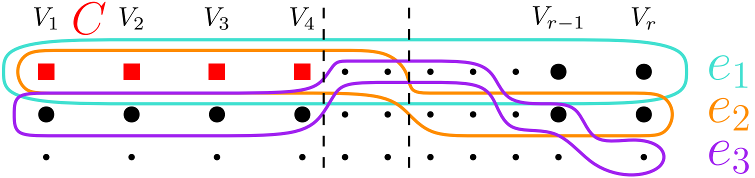

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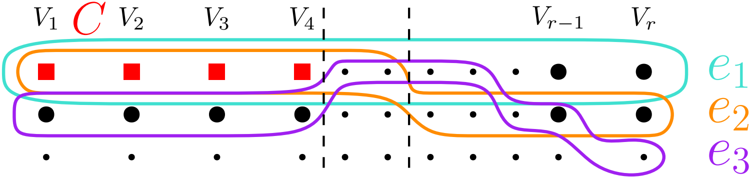

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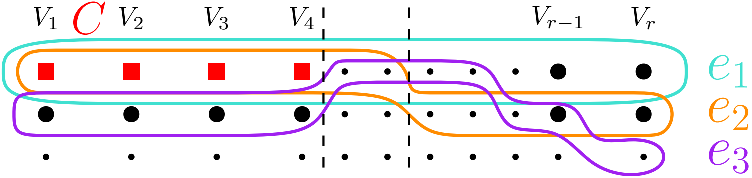
- $X \leq 2t - 1 \Rightarrow$ contradiction
- $X \leq \ 0 + 2(t-s) + (r-t)$
  $= 2(t-1) + (r - t - 2\left\lfloor \frac{r-t}{2} \right\rfloor) \leq 2t - 1.$



An application

k-wise intersecting hypergraphs

A hypergraph $\mathcal{H}$ is k-wise t-intersecting if for any $e_1, e_2, \ldots, e_k \in E(\mathcal{H})$ we have

$$|\cap_{i=1}^{k} e_i| \geq t.$$

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Theorem (Bishnoi-Das-M.-Szabó 2020+)

Let $\mathcal{H}$ be an r-uniform r-partite k-wise t-intersecting hypergraph. If $k \geq 3$ and $t \geq 1$, or $k = 2$ and $t > \frac{r}{3}$, then

$$\tau(\mathcal{H}) \leq \left\lfloor \frac{r-t}{k} \right\rfloor + 1,$$

and this bound is best possible.