List homomorphism problem parameterized by cutwidth

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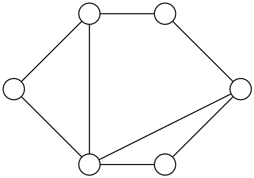
joint work with Paweł Rzążewski

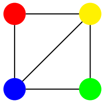
Homomorphisms & list homomorphisms

- fixed graph H
- given graph G

- we are looking for
  $f : V(G) \to V(H)$, s. t.:
  $uv \in E(G) \Rightarrow f(u)f(v) \in E(H)$



G



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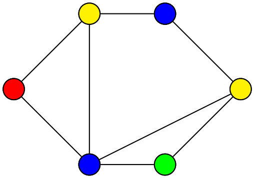

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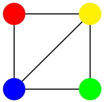
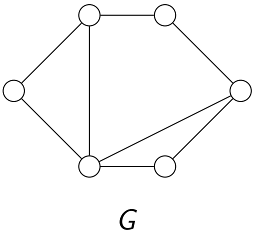
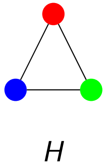


H

We denote the above problem by $\textsc{Hom}(H)$.

Homomorphisms & list homomorphisms





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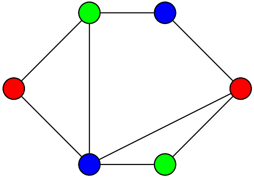
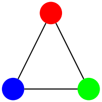
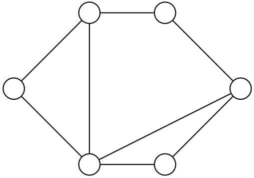
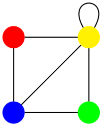




If $H \simeq K_k$, then homomorphism from G to H can be seen as k-COLORING of G.

Homomorphisms & list homomorphisms





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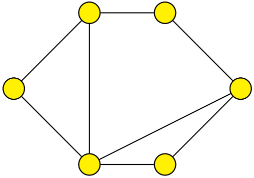
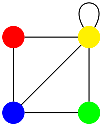
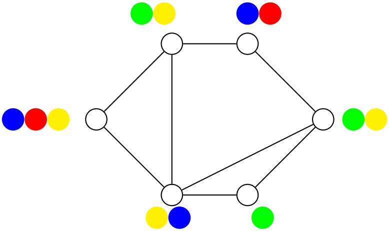




If H contains a vertex v with a loop, then we can map all vertices to v.

Homomorphisms & list homomorphisms

- G is given together with lists $L(v)$ of vertices of H
- we are looking for a homomorphism $f : V(G) \to V(H)$, s. t.: $f(v) \in L(v)$ for every v



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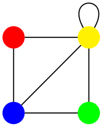

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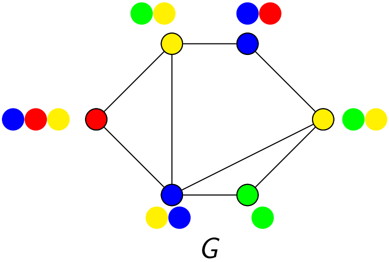
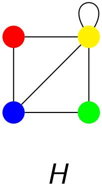


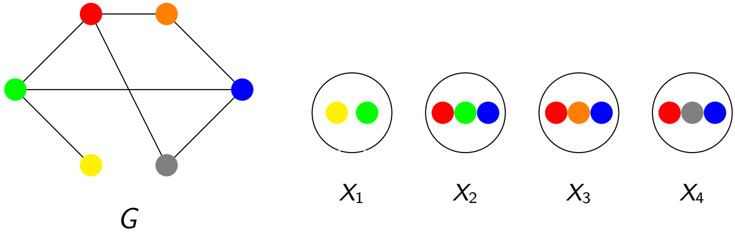


We denote the above problem by $\textsc{LHom}(H)$.

Complexity of coloring and homomorphism problems

- k-Coloring:
  - polynomial for $k = 1, 2$,
  - NP-complete for $k \geqslant 3$,
- Hom(H):
  - polynomial if H contains a loop or H is bipartite,
  - NP-complete otherwise (Hell, Nešetřil, 1990)
- LHom(H):
  - polynomial if H is a so-called *bi-arc* graph
    (if H is bipartite, bi-arc=complement of a circular-arc graph,
    and if H is reflexive, i.e., every vertex has a loop,
    bi-arc=interval graph)
  - NP-complete otherwise (Feder, Hell, Huang, 2003)

Parameters: pathwidth



Each edge must be contained in at least one bag and every vertex must be contained in consecutive bags.

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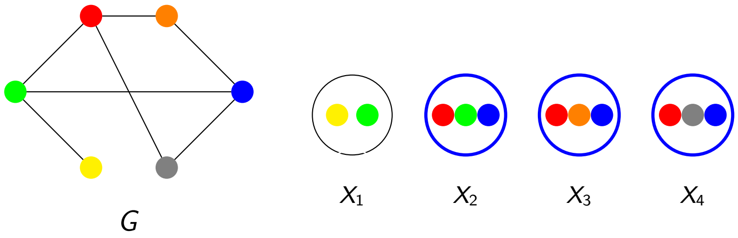

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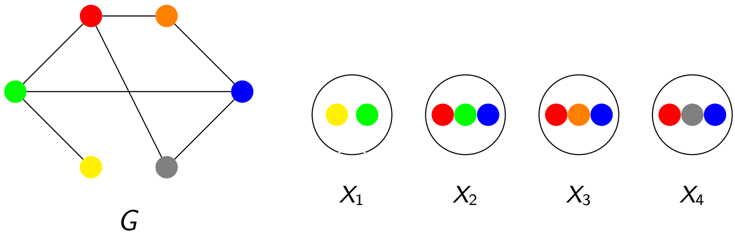
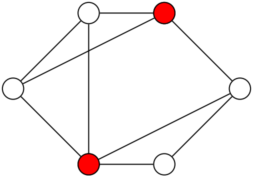


Each edge must be contained in at least one bag and every vertex must be contained in consecutive bags.
Width of decomposition $= \max |X_i| - 1$.
Pathwidth (denoted by $\mathrm{pw}(G)$)= minimum width among all decompositions.

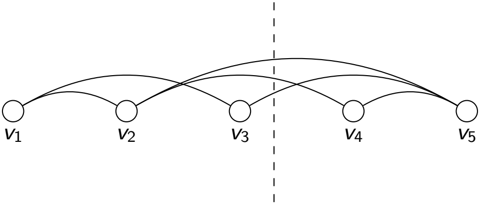
Parameters: size of a minimum feedback vertex set



Feedback vertex set – set F of vertices in G such that $G - F$ is a forest.

$\mathrm{fvs}(G)$ = size of a minimum feedback vertex set in G.

Parameters: cutwidth



Width of a linear ordering of vertices = maximum number of edges crossing any cut.
Cutwidth (denoted by $\mathrm{ctw}(G)$) = minimum width among all possible linear orderings.

Comparison of parameters

$$
\begin{tikzcd}
& {|V(G)|} & & ctw(G) \\
fvs(G) \arrow[ur] & & pw(G) \arrow[ul] \arrow[ur] & \\
& tw(G) \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

Hierarchy of all parameters considered in the talk.

The lack of an arrow which does not follow from the transitivity, denotes that two parameters are incomparable.

Complexity of k-Coloring

Can be solved in time:

- $k^{tw(G)} \cdot n^{\mathcal{O}(1)}$,
- $k^{pw(G)} \cdot n^{\mathcal{O}(1)}$,
- $k^{fvs(G)} \cdot n^{\mathcal{O}(1)}$,

But assuming the SETH* not in:

- $(k-\varepsilon)^{tw(G)} \cdot n^{\mathcal{O}(1)}$, ★
- $(k-\varepsilon)^{pw(G)} \cdot n^{\mathcal{O}(1)}$, ★
- $(k-\varepsilon)^{fvs(G)} \cdot n^{\mathcal{O}(1)}$. ★

★ (Lokshtanov,Marx,Saurabh,2018)

* Strong Exponential Time Hypothesis (SETH) $\Rightarrow$ for any $\varepsilon > 0$, CNF-SAT with n variables and m clauses cannot be solved in time $(2-\varepsilon)^n \cdot (n+m)^{\mathcal{O}(1)}$.

Complexity of k-Coloring

k-Coloring can be solved in time:

- $2^n \cdot n^{\mathcal{O}(1)}$, (Björklund, Husfeldt, Koivisto, 2009)
- $2^{\omega \cdot \mathrm{ctw}(G)} \cdot n^{\mathcal{O}(1)}$, (Jansen, Nederlof, 2019)

Is it possible to extend these algorithms to homomorphisms?

Assuming the ETH, there is no constant c such that Hom(H) can be solved in time $c^n \cdot n^{\mathcal{O}(1)}$ for every H. (Cygan *et al.*, 2017)

Question (Jansen)

Is there a constant c such that Hom(H) can be solved in time $c^{\mathrm{ctw}(G)} \cdot n^{\mathcal{O}(1)}$ for every H?

Complexity of $\textsc{LHom}(C_{2k})$

From now on we will assume that $H \simeq C_{2k}$ for $k \geqslant 5$.

For such H we define:

$$mim^*(H) := \text{size of a maximum induced matching in } H,$$

which for C_{2k} is actually equal to $\lfloor \frac{2k}{3} \rfloor$.

We will show that $\textsc{LHom}(H)$ cannot be solved in time $(mim^*(H) - \varepsilon)^{\frac{\mathrm{ctw}(G)}{3}} \cdot n^{\mathcal{O}(1)}$ for any $\varepsilon > 0$, unless the SETH fails.

Complexity of $\textsc{LHom}(C_{2k})$ cont.

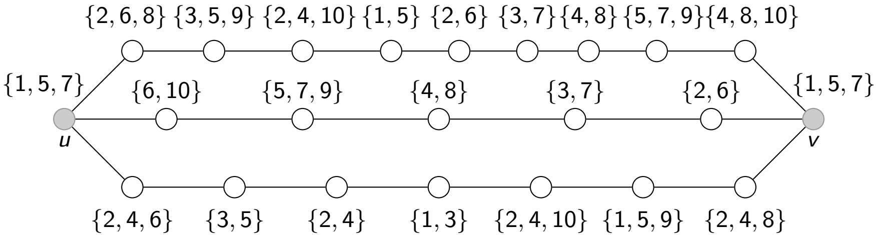
The main idea of the proof is to use the inequality:

$$\text{ctw}(G) \leqslant \text{pw}(G) \cdot \Delta(G) \text{ (Chung, Seymour, 1989)}$$

We will reduce from k-$\textsc{Coloring}$ – recall that it cannot be solved in time $(k-\varepsilon)^{\text{pw}(G)} \cdot n^{\mathcal{O}(1)}$, unless the SETH fails (Lokshtanov, Marx, Saurabh,2018)
Let $k := mim^*(H)$ (note that $k \geqslant 3$) and G be an instance of k-$\textsc{Coloring}$ given with a path decomposition $\mathcal{P}$ of width p.
Let M be an induced matching of size $mim^*(H)$ in H and let S, S' be the set of endpoints of M in each bipartition class of H.

Complexity of $\textsc{LHom}(C_{2k})$ cont.

We introduce an inequality gadget on S. Below an example for C_{10} with consecutive vertices $\{1, \ldots, 10\}$ and $S = \{1, 5, 7\}$.



In every list homomorphism mapping u, v to the same vertex is forbidden but every other pair is possible.

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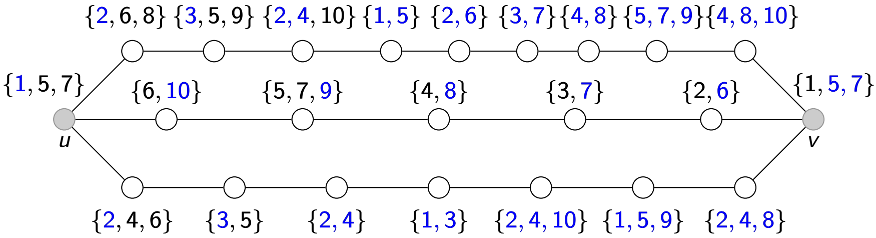

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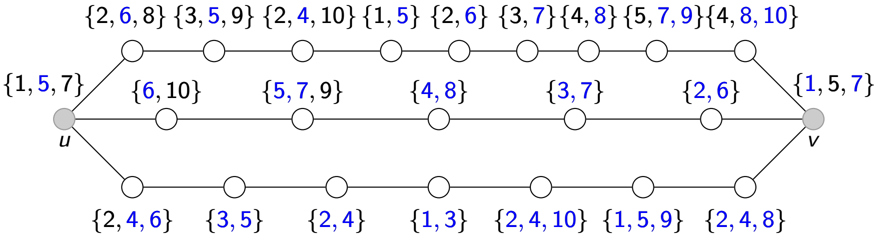

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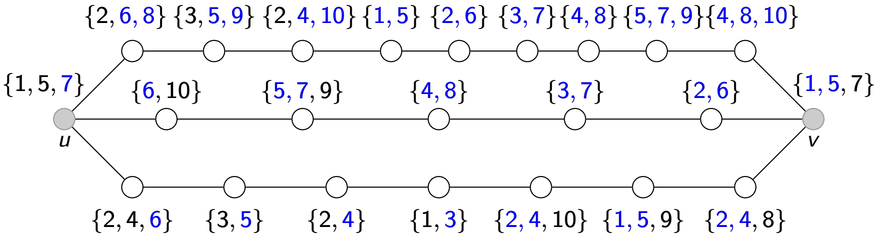
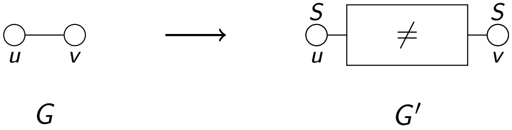


In every list homomorphism mapping u, v to the same vertex is forbidden but every other pair is possible.

Complexity of $\textsc{LHom}(C_{2k})$ cont.

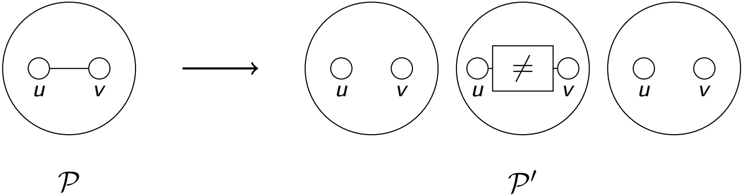
We construct an equivalent instance (G', L') of $\textsc{LHom}(H)$ by introducing an inequality gadget on every edge in G.



Let us mention that in every inequality gadget, not only the one for C_{10}, the degree of every vertex except u, v is at most 3.

Complexity of $\textsc{LHom}(C_{2k})$ cont.

We can easily modify path decomposition $\mathcal{P}$ of G of width p into a path decomposition $\mathcal{P}'$ of G' of width $p + g(H)$, where g is some function of H.

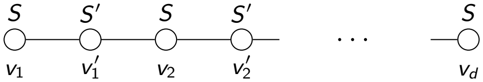


It only remains to reduce the degree while not increasing the pathwidth.

Complexity of $\textsc{LHom}(C_{2k})$ cont.

The only vertices, whose degree might be larger than 3 are original vertices of G. All of them are non-adjacent in G' and all have lists S.

Let v be a vertex of degree $d \geqslant 3$ in G'. We replace v with:



The fact that $S \cup S'$ induces a matching in H forces that in any list homomorphism every v_i must be mapped to the same vertex.

We connect each v_i with one neighbor of v in order of appearing in the path decoposition $\mathcal{P}'$.

The resulting instance $(\widetilde{G}, \widetilde{L})$ of $\textsc{LHom}(H)$ is equivalent to the instance (G', L') of $\textsc{LHom}(H)$. The maximum degree of $\widetilde{G}$ is 3.

Complexity of $\textsc{LHom}(C_{2k})$ cont.

Let us show that $\mathrm{pw}(\widetilde{G}) \leqslant p + g(H) + 1$.
In every bag of $\mathcal{P}'$ we replace v with v_1. Then for a bag X containing v_i and its neighbor in $\widetilde{G}$, we insert just after X bags:

$$X \cup \{v_i'\} \qquad X \setminus \{v_i\} \cup \{v_i', v_{i+1}\} \qquad X \setminus \{v_i\} \cup \{v_{i+1}\}$$

and we replace v_i with v_{i+1} in all following bags.
For every v_i, we choose a bag that does not already contain an extra vertex.

Complexity of $\textsc{LHom}(C_{2k})$ cont.

Now if $\textsc{LHom}(H)$ can be solved in time

$$(mim^*(H) - \varepsilon)^{\frac{\mathsf{ctw}(\widetilde{G})}{3}} \cdot n^{\mathcal{O}(1)},$$

then by our reduction and by inequality:

$$\mathsf{ctw}(\widetilde{G}) \leqslant \Delta(\widetilde{G}) \cdot \mathsf{pw}(\widetilde{G}) = 3\mathsf{pw}(\widetilde{G}),$$

$mim^*(H)$-$\textsc{Coloring}$ can be solved in time:

$$(mim^*(H) - \varepsilon)^{\mathsf{pw}(G)} \cdot n^{\mathcal{O}(1)},$$

which contradicts the SETH.

Our results

- By more careful reduction from SAT we can actually obtain lower bound $(mim^*(H) - \varepsilon)^{\mathsf{ctw}(G)} \cdot n^{\mathcal{O}(1)}$,
- We can extend our result to every "hard" graph H, i.e., such that $\textsc{LHom}(H)$ is NP-complete (the definition of $mim^*(H)$ is slightly different for general graphs).
- We provided similar lower bounds with weaker assumption: there is $\delta > 0$ such that for every "hard" H, there is no algorithm solving $\textsc{LHom}(H)$ in time $mim^*(H)^{\delta \cdot \mathsf{ctw}(G)} \cdot n^{\mathcal{O}(1)}$, unless the ETH fails.

Our results & questions

- Finally, we are able to extend all above results to the non-list version of the problem, in the case that H is a so-called *projective core* (for example every odd cycle is a projective core).
- For an odd cycle of length k we have $mim^*(H) = \lfloor \frac{2k}{3} \rfloor$ and thus the ETH-lower bound is: $\lfloor \frac{2k}{3} \rfloor^{\delta \cdot \mathrm{ctw}(G)} \cdot n^{\mathcal{O}(1)}$, so there is no constant c such that for every odd cycle C_k the $\textsc{Hom}(H)$ can be solved in time $c^{\delta \cdot \mathrm{ctw}(G)} \cdot n^{\mathcal{O}(1)}$, unless the ETH fails.

We also provided two algorithms for $\textsc{LHom}(H)$ parameterized by cutwidth, but their running time can be arbitrarily larger than our lower bound.

Question: Can our lower bound be improved? Or, is there a better algorithm?