Stability of extremal connected hypergraphs avoiding Berge-paths

Balázs Patkós

Alfréd Rényi Institute of Mathematics & Moscow Institute of Physics and Technology
joint work with D. Gerbner, D. Nagy, N. Salia, M. Vizer

8th Polish Combinatrial Conference, Spetember 16, 2020

Extremal problem: find largest structure size with prescribed property

Uniqueness: identify all structures achieving maximum size

Stability: all almost extremal structures are similar to extremal ones.

How to measure similarity?

STABILITY I

Theorem (Erdős, Simonovits)

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if G is an n-vertex K_{r+1}-free graph on n vertices with at least $(1 - \frac{1}{r} + \delta)\binom{n}{2}$ edges, then for some appropriately chosen Turán graph $T_{n,r}$, we have $|E(G) \triangle E(T_{n,r})| \leq \varepsilon n^2$.

Both addition and removal of edges are used as can be seen by considering almost balanced complete r-partite graphs.

STABILITY II

Theorem (Hilton, Milner)

Let $2k+1 \leq n$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is an intersecting family with $\cap_{F \in \mathcal{F}} F = \emptyset$, then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{h-1} + 1$.

That is, if $\mathcal{F}$ is a k-uniform intersecting family of size larger than $\binom{n-1}{k-1} - \binom{n-k-1}{h-1} + 1$, then $\mathcal{F}$ is a subfamily of an extremal family (a star). (No removal needed.)

The aim of this talk is to present a stability result of the second type on r-uniform connected hypergraphs avoiding paths of length k.

Graphs avoiding paths and cycles

Theorem (Erdős and Gallai)

For any $n \geq k \geq 1$, $ex(n, P_k) \leq \frac{(k-1)n}{2}$.
For any $n \geq k \geq 3$, $ex(n, \mathcal{C}_{\geq k}) \leq \frac{(k-1)(n-1)}{2}$.

The bounds are sharp for paths, if k divides n, and sharp for cycles, if $k-1$ divides $n-1$. These are shown by the example of n/k pairwise disjoint k-cliques for the path P_k, and adding an extra vertex joined by an edge to every other vertex for the class $\mathcal{C}_{\geq k+2}$ of cycles.

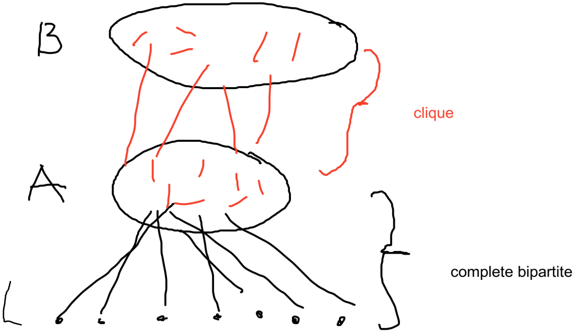
Later Faudree and Schelp gave the exact value of $ex(n, P_k)$ for every n.

Kopylov and independently Balister, Győri, Lehel, and Schelp determined the maximum number of edges $ex^{conn}(n, P_k)$ that an n-vertex connected graph can have without containing a path of length k.

The stability version of these results was proved by Füredi, Kostochka and Verstraëte.

Definition

For $n \geq k$ and $\frac{k}{2} > a \geq 1$ we define the graph $H_{n,k,a}$ as follows. The vertex set of $H_{n,k,a}$ is partitioned into three disjoint parts A, B and L such that $|A| = a$, $|B| = k - 2a$ and $|L| = n - k + a$. The edge set of $H_{n,k,a}$ consists of all the edges between L and A and also all the edges in $A \cup B$.



Definition

For $n \geq k$ and $\frac{k}{2} > a \geq 1$ we define the graph $H_{n,k,a}$ as follows. The vertex set of $H_{n,k,a}$ is partitioned into three disjoint parts A, B and L such that $|A| = a$, $|B| = k - 2a$ and $|L| = n - k + a$. The edge set of $H_{n,k,a}$ consists of all the edges between L and A and also all the edges in $A \cup B$.

Theorem (Füredi, Kostochka, Verstraëte)

Let $t \geq 2$, $n \geq 3t - 1$ and $k \in \{2t, 2t+1\}$. Suppose we have an n-vertex connected graph G with more edges than $|H_{n+1,k+1,t-1}| - n$ that does not contain any path with k edges. Then we have either

- *$k = 2t$, $k \neq 6$ and G is a subgraph of $H_{n,k,t-1}$, or*
- *$k = 2t+1$ or $k = 6$, and $G \setminus A$ is a star forest for $A \subseteq V(G)$ of size at most $t - 1$.*

Hypergraph paths and cycles (the definitions for today)

Definition

A *Berge-path* of length t is an alternating sequence of $t+1$ distinct vertices and t distinct hyperedges of the hypergraph, $v_1, e_1, v_2, e_2, v_3, \dots, e_t, v_{t+1}$ such that $v_i, v_{i+1} \in e_i$, for $i \in [t]$. The vertices $v_1, v_2, \dots, v_{t+1}$ are called *defining vertices* and the hyperedges $e_1, e_2, \dots, e_t$ are called *defining hyperedges* of the Berge-path. We denote the set of all Berge-paths of length t by $\mathcal{B}P_t$.

Similarly, a *Berge-cycle* of length t is an alternating sequence of t distinct vertices and t distinct hyperedges of the hypergraph, $v_1, e_1, v_2, e_2, v_3, \dots, v_t, e_t$, such that $v_i, v_{i+1} \in e_i$, for $i \in [t]$, where indices are taken modulo t. The vertices $v_1, v_2, \dots, v_t$ are called *defining vertices* and the hyperedges $e_1, e_2, \dots, e_t$ are called *defining hyperedges* of the Berge-cycle.

The study of the Turán numbers $ex_r(n, \mathcal{B}P_k)$ was initiated by Győri, Katona and Lemons, who determined the quantity in almost every case.

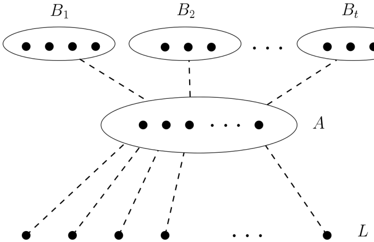
Later Davoodi, Győri, Methuku and Tompkins settled the missing case $r = k + 1$.

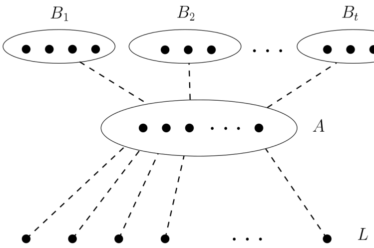
Analogously to graphs, a hypergraph is *connected*, if for any two of its vertices, there is a Berge-path containing both vertices.

For integers $n, a \geq 1$ and $b_1, \ldots, b_t \geq 2$ with $n \geq 2a + \sum_{i=1}^{t} b_i$ let us denote by $\mathcal{H}_{n,a,b_1,b_2,\ldots,b_t}$ the following r-uniform hypergraph.

- Let the vertex set of $\mathcal{H}_{n,a,b_1,b_2,\ldots,b_t}$ be $A \cup L \cup \bigcup_{i=1}^{t} B_i$, where $A, B_1, B_2, \ldots, B_t$ and L are pairwise disjoint sets of sizes $|A| = a$, $|B_i| = b_i$ $(i = 1, 2, \ldots, t)$ and $|L| = n - a - \sum_{i=1}^{t} b_i$.
- Let the hyperedges of $\mathcal{H}_{n,a,b_1,b_2,\ldots,b_t}$ be

$$\binom{A}{r} \cup \bigcup_{i=1}^{t} \binom{A \cup B_i}{r} \cup \left\{ \{c\} \cup A' : c \in L, A' \in \binom{A}{r-1} \right\}.$$





Observe that the number of hyperedges in $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ is

$$\left(n-a-\sum_{i=1}^{t} b_i\right)\binom{a}{r-1}+\sum_{i=1}^{t}\binom{a+b_i}{r}-(t-1)\binom{a}{r}.$$

Note that, if $a \leq a'$ and $b_i \leq b'_i$ for all $i = 1, 2, \dots, t$, then $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ is a subhypergraph of $\mathcal{H}_{n,a',b'_1,b'_2,\dots,b'_t}$. Finally, the length of the longest path in $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ is $2a - t + \sum_{i=1}^{t} b_i$ if $t \leq a+1$, and $a - 1 + \sum_{i=1}^{a+1} b_i$ if $t > a+1$ and the b_i's are in non-increasing order.

Theorem (Győri, Salia, Zamora and Füredi, Kostochka, Luo)

For all integers k, r *with* $k \geq 2r + 13 \geq 18$ *there exists* $n_{k,r}$ *such that if* $n > n_{k,r}$*, then we have*

- $ex_r^{conn}(n, \mathcal{B}P_k) = |\mathcal{H}_{n,\lfloor \frac{k-1}{2} \rfloor}|$*, if* k *is odd, and*
- $ex_r^{conn}(n, \mathcal{B}P_k) = |\mathcal{H}_{n,\lfloor \frac{k-1}{2} \rfloor,2}|$*, if* k *is even.*

Depending on the parity of k*, the unique extremal hypergraph is* $\mathcal{H}_{n,\lfloor \frac{k-1}{2} \rfloor}$ *or* $\mathcal{H}_{n,\lfloor \frac{k-1}{2} \rfloor,2}$.

Our aim is to obtain a result stating that if $\mathcal{H}$ is connected, without a Berge-path of length k, and contains many hyperedges, then $\mathcal{H}$ is a subhypergraph of an extremal one.

We have to come up with a "second best" solution. This is going to be $\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,3}$ if k is odd and $\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,4}$ if k is even.

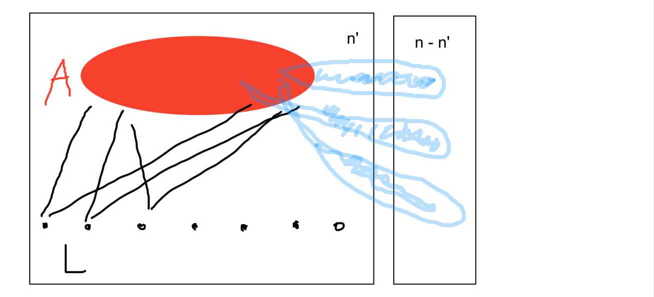
Our aim is to obtain a result stating that if $\mathcal{H}$ is connected, without a Berge-path of length k, and contains many hyperedges, then $\mathcal{H}$ is a subhypergraph of an extremal one.

We have to come up with a "second best" solution. This is going to be $\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,3}$ if k is odd and $\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,4}$ if k is even.

So an extremal hypergraph contains roughly $\binom{\lfloor\frac{k-1}{2}\rfloor}{r-1}n$ hyperedges, while second best contains roughly $\binom{\lfloor\frac{k-3}{2}\rfloor}{r-1}n$ hyperedges.

Problem I

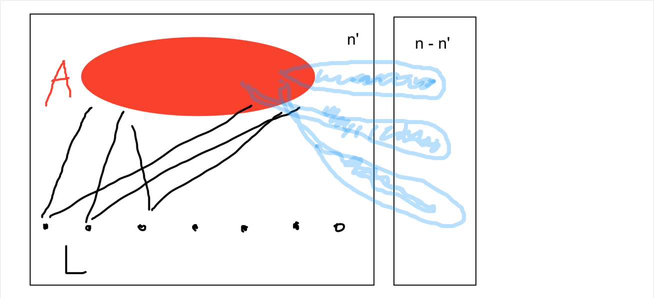
One can start with a $\mathcal{H}_{n',\lfloor\frac{k-1}{2}\rfloor}$ for some n' slightly smaller than n and add hyperedges that partition $[n'+1, n'+2, \dots, n]$ and contains only vertices from "the A part" of $\mathcal{H}_{n',\lfloor\frac{k-1}{2}\rfloor}$



If $n' = n - C$, then we lose just $C\binom{\lfloor\frac{k-1}{2}\rfloor}{r-1}$ hyperedges and these hypergraphs are not subhypergraphs of $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor,2}$.

Problem I

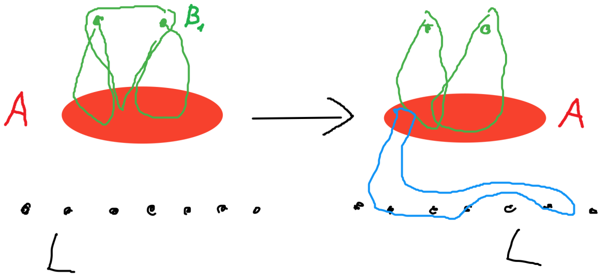
One can start with a $\mathcal{H}_{n',\lfloor\frac{k-1}{2}\rfloor}$ for some n' slightly smaller than n and add hyperedges that partition $[n'+1, n'+2, \ldots, n]$ and contains only vertices from "the A part" of $\mathcal{H}_{n',\lfloor\frac{k-1}{2}\rfloor}$



If $n' = n - C$, then we lose just $C\binom{\lfloor\frac{k-1}{2}\rfloor}{r-1}$ hyperedges and these hypergraphs are not subhypergraphs of $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor,2}$.
Every problematic edge contains a vertex of degree 1. So hopefully our aimed theorem is true for hypergraphs with minimum degree 2.

Problem II

If k is even and one removes the hyperedges of $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor,2}$ that contain both vertices of B_1, but adds any hyperedge that contains at least 3 vertices from outside "the A part of $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor,2}$, the length of the longest path remains the same. Or equivalently, one starts with $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor}$ and adds one hyperedge that contains at least 3 vertices from outside "the A part of $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor}$. Let us denote these hypergraphs by $\mathcal{H}^{+}_{n,\lfloor\frac{k-1}{2}\rfloor}$.



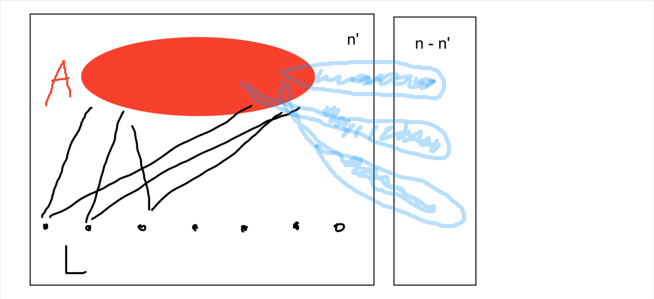
The min degree at least 2 result

Theorem (GNPSV, 20+)

For any $\varepsilon > 0$ there exist integers $q = q_\varepsilon$ and $n_{k,r}$ such that if $r \geq 3$, $k \geq (2+\varepsilon)r + q$, $n \geq n_{k,r}$ and $\mathcal{H}$ is a connected n-vertex, r-uniform hypergraph with minimum degree at least 2, without a Berge-path of length k, then we have the following.

- *If k is odd and $|\mathcal{H}| > |\mathcal{H}_{n,\frac{k-3}{2},3}| = (n - \frac{k+3}{2})\binom{\frac{k-3}{2}}{r-1} + \binom{\frac{k+3}{2}}{r}$, then $\mathcal{H}$ is a subhypergraph of $\mathcal{H}_{n,\frac{k-1}{2}}$.*
- *If k is even and $|\mathcal{H}| > |\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,4}| = (n - \lfloor\frac{k+5}{2}\rfloor)\binom{\lfloor\frac{k-3}{2}\rfloor}{r-1} + \binom{\lfloor\frac{k+5}{2}\rfloor}{r}$, then $\mathcal{H}$ is a subhypergraph of $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor,2}$ or $\mathcal{H}^+_{n,\lfloor\frac{k-1}{2}\rfloor}$.*

The general result.



Let $\mathbb{H}'_{n,a,b_1,b_2,\dots,b_t}$ be the class of hypergraphs that can be obtained from $\mathcal{H}_{n',a,b_1,b_2,\dots,b_t}$ for some $n' \leq n$ by adding hyperedges of the form $A'_j \cup D_j$, where the D_j's partition $[n'] \setminus [n]$, all D_j's are of size at least 2 and $A'_j \subseteq A$ for all j. Let us define $\mathbb{H}^+_{n',\lfloor \frac{k-1}{2} \rfloor}$ analogously.

Theorem (GNPSV, 20+)

For any $\varepsilon > 0$ there exist integers $q = q_\varepsilon$ and $n_{k,r}$ such that if $r \geq 3$, $k \geq (2+\varepsilon)r + q$, $n \geq n_{k,r}$ and $\mathcal{H}$ is a connected n-vertex, r-uniform hypergraph without a Berge-path of length k, then we have the following.

- *If k is odd and $|\mathcal{H}| > |\mathcal{H}_{n,\frac{k-3}{2},3}|$, then $\mathcal{H}$ is a subhypergraph of some $\mathcal{H}' \in \mathbb{H}'_{n,\frac{k-1}{2}}$.*
- *If k is even and $|\mathcal{H}| > |\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,4}|$, then $\mathcal{H}$ is a subhypergraph of some $\mathcal{H}' \in \mathbb{H}'_{n,\lfloor\frac{k-1}{2}\rfloor,2}$ or $\mathbb{H}^+_{n,\lfloor\frac{k-1}{2}\rfloor}$.*

Skeleton of proof

A reduction:

starting from $\mathcal{H}$ applying a standard greedy process, one can obtain an induced subhypergraph $\mathcal{H}^* \subseteq \mathcal{H}$ with the extra property:

for every $U \subseteq V(\mathcal{H}^*)$ with $|U| \leq k/2$, the number of hyperedges of $\mathcal{H}^*$ meeting U is at least $|U|\binom{\lfloor \frac{k-3}{2} \rfloor}{r-1}$.

In particular, every degree is at least $\binom{\lfloor \frac{k-3}{2} \rfloor}{r-1}$.

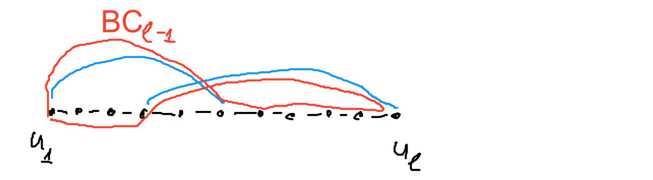
Preliminaries for the special case: get some structure of $\mathcal{H}$:

Claim

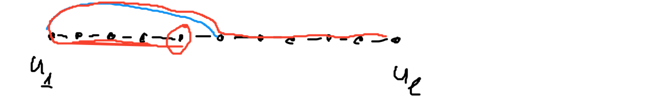
Let $\ell - 1$ be the length of the longest Berge-path in $\mathcal{H}$. Then $\ell \geq k - 3$ and $\mathcal{H}$ contains a Berge-cycle of length $\ell - 1$.

Take a longest path $u_1, e_1, u_2, e_2, \ldots, e_{\ell-1}, u_\ell$. Let $\mathcal{F} = \{e_1, e_2, \ldots, e_{\ell-1}\}$.

- $N_{\mathcal{H}\setminus\mathcal{F}}(u_1)$ and $N_{\mathcal{H}\setminus\mathcal{F}}(u_\ell)$ must "differ"



- Lots of possible left endpoints of longest path.



- The set U of left endpoints will not have the set degree property

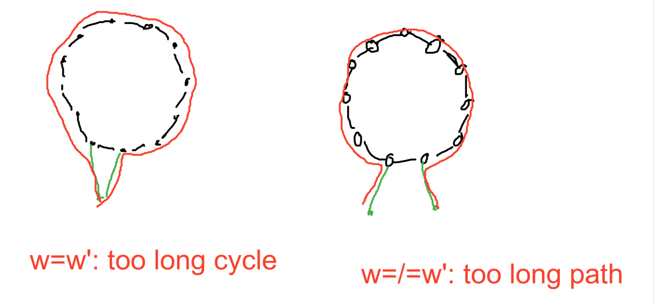
C is the longest cycle from previous claim. V is the set of its defining vertices.

For any $w \in V(\mathcal{H}) \setminus V$ we let

D_w be the set of $v \in V$ such that there exists at least 2 hyperedges $h_1, h_2 \in \mathcal{H} \setminus C$ with $w, v \in h_i$.
(the double neighborhood of w on C)

Properties of D_w:

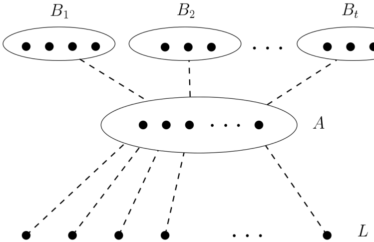
- $|D_w| \geq \left\lfloor \frac{k-3}{2} \right\rfloor$
- for any w, w' ($w = w'$ included) the double neighborhoods cannot contain consecutive vertices of C



Lengthy case analysis according to:

- length of C which can be $2\left\lfloor\frac{k-3}{2}\right\rfloor, 2\left\lfloor\frac{k-3}{2}\right\rfloor+1, 2\left\lfloor\frac{k-3}{2}\right\rfloor+2$ or $2\left\lfloor\frac{k-3}{2}\right\rfloor+3$
- are there any pair w, w' with both $D_w \setminus D_{w'}$ and $D_{w'} \setminus D_w$ non-empty?
- does there exist w with $|D_w| \geq \left\lfloor\frac{k-1}{2}\right\rfloor$?

In all cases, we try to define an embedding $\mathcal{H} \subseteq \mathcal{H}_{n,a,b_1,b_2,\dots,b_s}$



with $A = D_w$ or $A = D_w \cup D_{w'}$.

Either get the conclusion of theorem (if $a = \left\lfloor \frac{k-1}{2} \right\rfloor$)

or get a contradiction with assumption on $|\mathcal{H}|$ (if $a < \left\lfloor \frac{k-1}{2} \right\rfloor$) .
Details are quite tedious.

Thank you for your attention!