On ordered Ramsey numbers of tripartite 3-uniform hypergraphs

Martin Balko and Máté Vizer

Alfréd Rényi Institute of Mathematics
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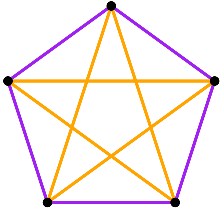

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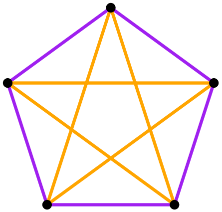
Example:  $\mathrm{R}(C_4) = 6$

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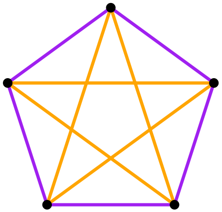
Example:  $\mathrm{R}(C_4) = 6$

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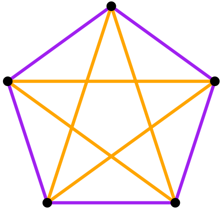

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- Classical bounds of Erdős and Szekeres: $2^{n/2} \leq \mathrm{R}(K_n) \leq 2^{2n}$ for $k = 2$.
- For $k \geq 3$, the Ramsey numbers are much less understood, for example:

$$2^{\Omega(n^2)} \leq R(K_n^{(3)}) \leq 2^{2^{O(n)}}.$$

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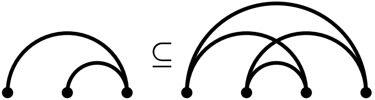

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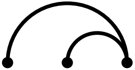
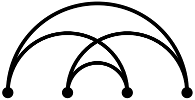
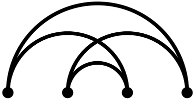
 $\subseteq$ 

 $\nsubseteq$ 

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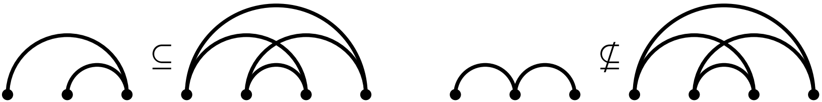

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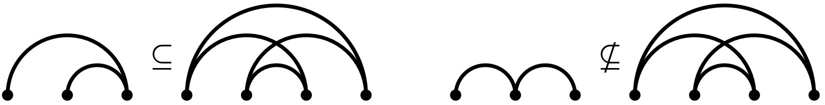

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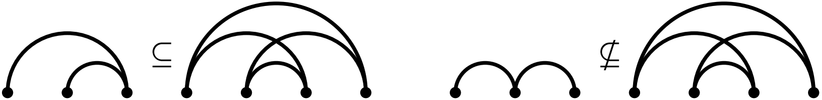

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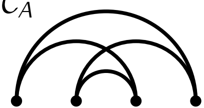

$\overline{\mathrm{R}}(\mathcal{C}_A) = 10$

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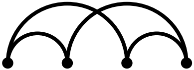

$\overline{\mathrm{R}}(\mathcal{C}_B) = 11$

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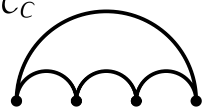

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- To obtain a polynomial upper bound on $\overline{\mathrm{R}}(\mathcal{G})$ we need to bound another parameter besides the maximum degree.

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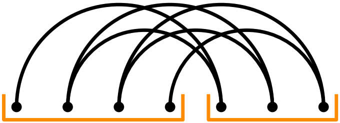

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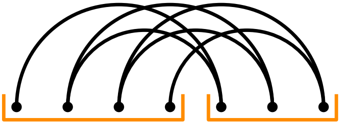

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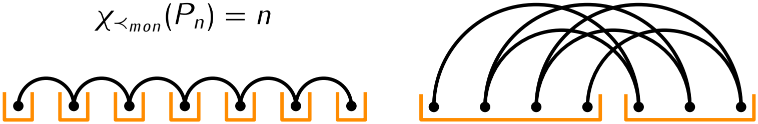

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- Stronger bound: $\overline{\mathrm{R}}(\mathcal{G}) \leq n^{O(k \log p)}$ (Conlon, Fox, Lee, and Sudakov).

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For every n,

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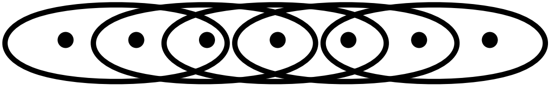

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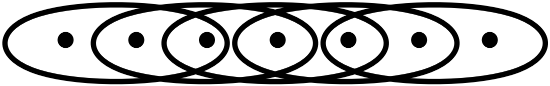

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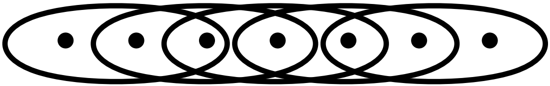

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Proposition 8

Let χ, k be integers with $\chi \geq k \geq 2$ and let $\mathcal{H}$ be an ordered k-graph on n vertices with interval chromatic number χ. Then there is a constant $c = c(\chi)$ such that

$$\overline{R}(\mathcal{H}) \leq 2^{c(n^{\chi-1})}.$$

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