Edge-distinguishing of star-free graphs

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Kraków, 14-18 September 2020

Definition

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Definition (Kalinowski, Pilśniakl, 2015)

The distinguishing index $D'(G)$ of a graph G is the least number d such that G admits an edge-colouring with d colours that breaks every nontrivial automorphism of G.

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- $D'(C_n) = 2$ for $n \geq 6$,

Preliminary results for basic classes of graphs

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- $D'(C_n) = 2$ for $n \geq 6$,
- $D'(K_n) = 2$ for $n \geq 6$,
- $D'(K_{1,n}) = n$ for $n \geq 2$.

In the first paper on the distinguishing index [1] the following result was proved.

Theorem (Kalinowski, Pilśniak 2015)

If G is a connected graph of order $n \geq 3$, then

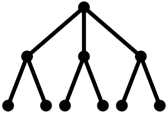
$$D'(G) \leq \Delta(G),$$

except for three small cycles C_3, C_4 and C_5.

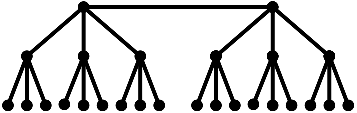
[1] R. Kalinowski and M. Pilśniak, *Distiguishing graphs by edge-colourings*, European J. Combin. 45(2015) 124 –131

The authors showed that this result is best possible by providing two classes of trees, i.e. the symmetric and bisymmetric trees, for which the equality in the bound holds. The examples of such trees are drawn below.

Example



a symmetric tree T
$\Delta(T) = 3, D'(T) = 3$



a bisymmetric tree T
$\Delta(T) = 4, D'(T) = 4$

Later Pilśniak improved this result [2].

Theorem (Pilśniak 2017)

Let G be a connected graph that is neither a symmetric nor a bisymmetric tree. If the maximum degree of G is at least 3, then

$$D'(G) \leq \Delta(G) - 1,$$

unless G in K_4 or $K_{3,3}$.

[2] M. Pilśniak, *Improving Upper Bounds for the Distinguishing Index*, Ars Math. Contemp. 13 (2017) 259–274

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This bound again is best possible in general. Therefore we focus our research on finding better upper bound for specific classes of graphs.

[2] M. Pilśniak, *Improving Upper Bounds for the Distinguishing Index*, Ars Math. Contemp. 13 (2017) 259–274

Traceable graphs

Theorem (Pilśniak 2017)

If G is a traceable graph of order $n \geq 7$, then

$$D'(G) \leq 2.$$

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Cartesian powers

Theorem (A. Gorzkowska, R. Kalinowski, M. Pilśniak 2017)

Let G be a connected graph and $k \geq 2$, then

$$D'(G^k) = 2,$$

with the only exception $D'(K_3^2) = 3$.

minimum degree condition

Theorem (Imrich, Kalinowski, Pilśniak, Woźniak 2019)

If G is a connected graph with minimum degree $\delta(G) \geq 2$, then

$$D'(G) \leq \left\lceil \sqrt{\Delta(G)} \right\rceil + 1.$$

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regular graphs

Theorem (F. Lehner, M. Pilśniak, M. Stawiski 2020)

Let G be a connected Δ-regular graph, then

$$D'(G) \leq 3.$$

Our research focuses on the class of claw-free graphs.

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We are motivated by the following result and conjecture of Pilśniak.

Theorem (Pilśniak 2017)

If G is a connected claw-free graph, then

$$D'(G) \leq 3.$$

Conjecture (Pilśniak 2017)

There exists a constant $n_0 \in \mathbb{N}_+$ such that every connected claw-free graph G of order $n \geq n_0$ satisfies

$$D'(G) \leq 2.$$

Not only do we prove the conjecture of Pilśniak, we extend this result to a more general class of graphs.

Definition

We say that a graph G is $K_{1,s}$-free if it does not contain a star $K_{1,s}$ as an induced subgraph.

[3] A. Gorzkowska, E. Kargul, S. Musiał and K. Pal, *Edge-distinguishing of star-free graphs*, Electron. J. Combin. 27(3) (2020) P3.30

Definition

We say that a graph G is $K_{1,s}$-free if it does not contain a star $K_{1,s}$ as an induced subgraph.

We call all $K_{1,s}$-free graphs, for $s \geq 3$, star-free graphs.

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Theorem (Gorzkowska, Kargul, Musiał, Pal 2020 [3])

Let G be a connected, $K_{1,s}$-free graph, with $s \geq 3$, of order at least 6, then

$$D'(G) \leq s - 1.$$

[3] A. Gorzkowska, E. Kargul, S. Musiał and K. Pal, *Edge-distinguishing of star-free graphs*, Electron. J. Combin. 27(3) (2020) P3.30

Proof.

The main idea of the proof is that we colour the edges of the graph G in a specific way and then we show that this colouring is indeed distinguishing. To find desired colouring we proceed as follows:

1. We pre-colour some specified induced subgraph H of the graph G.
2. We use an ALGORITHM to colour the remaining edges of the graph G.

We treat the case $s = 3$ separately from the cases $s \geq 4$; a different subgraph H is chosen for pre-colouring.

The proprieties of the ALGORITHM and the structure of star-free graphs are used in the proof that the returned colouring is distinguishing.

$\square$

Our result is best possible, in the sense that there exist classes of $K_{1,s}$ - free graphs for which $D'(G) = s - 1$. Namely:

1. the star $K_{1,s-1}$,
2. symmetric and bisymmetric trees of maximal degree $s - 1$.

Thank you for reading!