Construction of prime 3-uniform hypergraphs.

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Motivation

Problem: BILT (2004)

Given a set V, find a necessary and sufficient condition for a family of subsets of cardinality 3 of V to be the 3-cycles of a tournament defined on V.

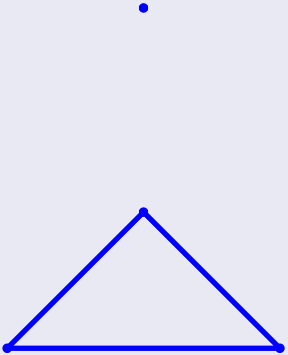
A. Boussaïri, P. Ille, G. Lopez, S. Thomassé
The C_3-structure of the tournaments
Discrete Math. 277 (2004) 29–43.

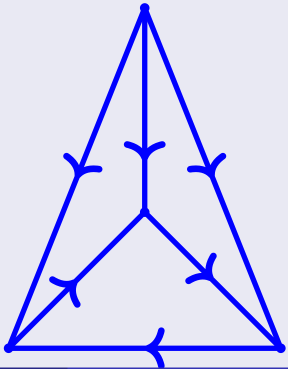
Hypergraph and its realization

Definition

Let H be a 3-uniform hypergraph. A tournament T, with the same vertex set as H, is a realization of H if the hyperedges of H are exactly the 3-subsets of T that induce 3-cycles.

Example: A Hypergraph H and its realisation T





Motivation

Theorem

Given a 3-uniform hypergraph H, then H is realizable if and only if all its prime, 3-uniform and induced subhypergraphs are realizable.

Theorem

Given a realizable 3-uniform hypergraph H, then H is prime if and only if its realizations are prime.

A. Boussaïri, B. Chergui, P. Ille, M. Zaidi
3-uniform hypergraphs: modular decomposition and realization by tournaments
Contrib. to Discrete Math. 15 (2020) 121-153.

Motivation

These results lead us to study the prime, 3-uniform and induced subhypergraphs of a prime 3-uniform hypergraph. Precisely, consider a prime 3-uniform hypergraph H. We are interesting in the integers $n \in \{3, ..., v(H) - 1\}$ for which there exists $W \subseteq V(H)$ satisfying $|W| = n$ and $H[W]$ is prime.

Module of a Hypergraph

Definition

Let H be a hypergraph. A subset M of $V(H)$ is a module of H if for each $e \in E(H)$ such that $e \cap M \neq \emptyset$ and $e \setminus M \neq \emptyset$, there exists $m \in M$ such that $e \cap M = \{m\}$, and for every $n \in M$, we have $(e \setminus \{m\}) \cup \{n\} \in E(H)$.

- $\emptyset$, $V(H)$ and $\{v\}$, where $v \in V(H)$, are modules of H, called trivial modules.
- A hypergraph H is indecomposable if all its modules are trivial, otherwise it is decomposable.
- A hypergraph H is prime if it is indecomposable, with $v(H) \geq 3$.

Study of the existence of prime subhypergraphs of a small size in a prime k-uniform hypergraph

Lemma

Let T be a strongly conected tournament (with $v(T) \geq 3$). For every $v \in V(T)$, there exists $X \subseteq V(T)$ such that

$$v \in X,\ |X| = 3, \text{ and } T[X] \text{ is prime.} \tag{1}$$

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Remark: An analogue of (1) is holds for prime tournaments.

Study of the existence of prime subhypergraphs of a small size in a prime k-uniform hypergraph

Lemma

Let H be a prime k-uniform hypergraph, where $k \geq 3$. For every $v \in V(H)$, there exists $X \subseteq V(H)$ such that

$$v \in X, \ |X| = k, \text{ and } H[X] \text{ is prime.} \tag{2}$$

Question : An analogue of (2) is holds for prime hypergraphs not necessarily k-uniform?

Study of the existence of prime subhypergraphs of a small size in a prime k-uniform hypergraph

Lemma

Let H be a prime k-uniform hypergraph, where $k \geq 3$. For every $v \in V(H)$, there exists $X \subseteq V(H)$ such that

$$v \in X, \ |X| = k, \text{ and } H[X] \text{ is prime.} \tag{2}$$

Question : An analogue of (2) is holds for prime hypergraphs not necessarily k-uniform?

Remark

The previous Lemma does not hold for a hypergraph which is not k-uniform for some $k \geq 3$. Consider a hypergraph H such that $E(H) = \{V(H)\}$ and $V(H) \geq 4$. We can show that H is prime. Nevertheless, for $X \subsetneq V(H)$ with $|X| \geq 3$, $H[X]$ is decomposable.

Construction of prime subhypergraphs of a larger size in a prime hypergraph

Notation

Let H be a hypergraph. Given $X \subsetneq V(H)$ such that $H[X]$ is prime, consider the following subsets of $V(H) \setminus X$.

- $Ext_H(X)$ denotes the set of $v \in V(H) \setminus X$ such that $H[X \cup \{v\}]$ is prime;
- $\langle X \rangle_H$ denotes the set of $v \in V(H) \setminus X$ such that X is a module of $H[X \cup \{v\}]$;
- for each $y \in X$, $X_H(y)$ denotes the set of $v \in V(H) \setminus X$ such that $\{y, v\}$ is a module of $H[X \cup \{v\}]$.

Construction of prime subhypergraphs of a larger size in a prime hypergraph

We use the set $p_{(H;X)}$ defined as follows.
$p_{(H;X)} := \{Ext_H(X), \langle X \rangle_H\} \cup \{X_H(y) : y \in X\}$

Lemma

Let H be a 3-uniform hypergraph. Consider $X \subsetneq V(H)$ such that $H[X]$ is prime. The set $p_{(H,X)}$ is a partition of $V(H) \setminus X$.

Construction of prime subhypergraphs of a larger size in a prime hypergraph

The next Theorem allows us to obtain prime subhypergraphs of a larger size in a prime 3-uniform hypergraph.

Theorem

Let H be a prime 3-uniform hypergraph. Consider $X \subsetneq V(H)$ such that $H[X]$ is prime. There exists $Y \subseteq V(H) \setminus X$ such that $1 \leq |Y| \leq 3$ and $H[X \cup Y]$ is prime.

Construction of prime subhypergraphs of a larger size in a prime hypergraph

Corollary

Let H be a prime 3-uniform hypergraph. If $v(H) \geq 4$, then there exists $Y \subseteq V(H)$ such $1 \leq |Y| \leq 3$ and $H - Y$ is prime.

Construction of prime subhypergraphs of a larger size in a prime hypergraph

Corollary

Let H be a prime 3-uniform hypergraph. If $v(H) \geq 4$, then there exists $Y \subseteq V(H)$ such $1 \leq |Y| \leq 3$ and $H - Y$ is prime.

This Corollary is obtained By using the previous Theorem several times and the following Lemma

Lemma

Let H be a prime k-uniform hypergraph, where $k \geq 3$. For every $v \in V(H)$, there exists $X \subseteq V(H)$ such that

$$v \in X,\ |X| = k, \text{ and } H[X] \text{ is prime.}$$

Construction of prime subhypergraphs of a larger size in a prime hypergraph

Corollary

Let H be a prime 3-uniform hypergraph. If $v(H) \geq 4$, then there exists $Y \subseteq V(H)$ such $1 \leq |Y| \leq 3$ and $H - Y$ is prime.

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We improve Corollary above by the following.

Construction of prime subhypergraphs of a larger size in a prime hypergraph

Corollary

Let H be a prime 3-uniform hypergraph. If $v(H) \geq 4$, then, there exist $v, w \in V(H)$ such that $H - \{v, w\}$ is prime.

Construction of prime subhypergraphs of a larger size in a prime hypergraph

Corollary

Let H be a prime 3-uniform hypergraph. If $v(H) \geq 4$, then, there exist $v, w \in V(H)$ such that $H - \{v, w\}$ is prime.

Remark

This Corollary does not hold if we require also that $v \neq w$, whereas it does for a prime tournament T such that $v(T) \geq 7$.

Construction of prime subhypergraphs of a larger size in a prime hypergraph

Corollary

Let H be a prime 3-uniform hypergraph. If $v(H) \geq 4$, then, there exist $v, w \in V(H)$ such that $H - \{v, w\}$ is prime.

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This Corollary does not hold if we require also that $v \neq w$, whereas it does for a prime tournament T such that $v(T) \geq 7$.

We propose the following problem.

Construction of prime subhypergraphs of a larger size in a prime hypergraph

Problem

Charaterize the prime 3-uniform hypergraphs H such that $H - \{v, w\}$ is decomposable for any distinct $\{v, w\} \subsetneq V(H)$.

A. Boussaïri, B. Chergui, P. Ille, M. Zaidi
Prime 3-uniform hypergraphs
submitted

Construction of prime subhypergraphs of a larger size in a prime hypergraph

Theorem (Ehrenfeuchtet et al. 1999)

Given a prime tournament T, consider $X \subseteq V(T)$ such that $T[X]$ is prime. If $|V(T) \setminus X| \geq 2$, then there exist $v, w \in V(T) \setminus X$ such that $v \neq w$ and $T[X \cup \{v, w\}]$ is prime.

A. Ehrenfeucht, T. Harju, G. Rozenberg,
The Theory of 2-Structures, A Framework for Decomposition and Transformation of Graphs
World Scientific, Singapore, 1999.

Construction of prime subhypergraphs of a larger size in a prime hypergraph

Theorem (Ehrenfeuchtet et al. 1999)

Given a prime tournament T, consider $X \subseteq V(T)$ such that $T[X]$ is prime. If $|V(T) \setminus X| \geq 2$, then there exist $v, w \in V(T) \setminus X$ such that $v \neq w$ and $T[X \cup \{v, w\}]$ is prime.

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Remark:

An analogue of this Theorem does not hold for prime 3-uniform hypergraphs and also for prime hypergraphs.

Construction of prime subhypergraphs of a larger size in a prime hypergraph

From the previous Remark, we conjecture the following.

Conjecture

Let H be a prime hypergraph. Consider $X \subsetneq V(H)$ such that $H[X]$ is prime. There exists $Y \subseteq V(H) \setminus X$ such that $H[X \cup Y]$ is prime, and $1 \leq |Y| \leq max(\{|e| : e \in E(H)\})$.

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Prime 3-uniform hypergraphs
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A positive answer is provided by the next theorem for prime 3-uniform hypergraph.

Construction of prime subhypergraphs of a larger size in a prime hypergraph

Theorem

Let H be a prime 3-uniform hypergraph. Consider $X \subsetneq V(H)$ such that $H[X]$ is prime. There exists $Y \subseteq V(H) \setminus X$ such that $1 \leq |Y| \leq 3$ and $H[X \cup Y]$ is prime.

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Thank you for your attention