Concentration of Extension Counts in $\mathbb{G}(n, p)$

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Based on paper *Counting extensions revisited* [arxiv:1911.03012]

Approximate regularity of $\mathbb{G}(n,p)$

- Random graph $\mathbb{G}(n,p)$: vertices $1,\ldots,n$, each edge included independently with probability p.
- We assume $n \to \infty$, allow $p = p(n)$
- Easy fact: if $np \gg \log n$, *all* degrees of $\mathbb{G}(n,p)$ are w.h.p. $(1+o(1))np$
- Why? Since $\deg(v) \sim \mathrm{Bin}(n-1,p)$, for small constant $\varepsilon > 0$

$$\mathbf{Pr}(\exists v : |\deg(v) - np| > \varepsilon np) \le n \exp\{-\Omega(\varepsilon^2 np)\}$$

The maximum degree of $\mathbb{G}(n, p)$

- For $G \sim \mathbb{G}(n, p)$,

$$\mu := \mathbf{E}\,\deg_G(v) \sim pn, \sigma^2 := \mathbf{Var}\,\deg_G(v) \sim p(1-p)n$$

  Easy fact: if $\mu \gg \log n$

$$\frac{\max_v \deg_G(v)}{\mu} \xrightarrow{\text{p}} 1$$

- More precise: [Bollobás '80] for $\sigma^2 \gg (\log n)^3$

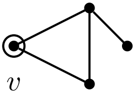
$$\frac{\max_{v \in [n]} \deg_G(v) - \mu}{\sigma\sqrt{2 \log n}} \xrightarrow{\text{p}} 1.$$

- Intuition: $\deg_G(v) \approx Z \sim \mathcal{N}(\mu, \sigma^2)$ and standard estimate

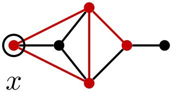
$$\mathbf{Pr}\left(Z \geq \mu + x\sigma\right) = \exp\left(-\frac{x^2}{2}(1 + o(1))\right), \qquad x \to \infty.$$

Generalization of vertex degrees: extension counts

- *Rooted graph* H: a graph H with a distinguished *root* $v \in V(H)$

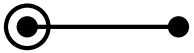


- H-*Extension* of $x \in V(G)$ is a copy of H with $v \mapsto x$.

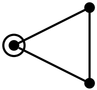


Extension counts in random graphs

- $X_H(x)$ = number of H-extensions of vertex x in $\mathbb{G}(n,p)$
- Examples:



$H = K_2$ rooted at one end:
$X_H(x) = \deg(x)$



$H = K_3$: $X_H(x)$ = number of triangles containing x

- Extends to graphs rooted on more vertices
- For example: if $H =$ ,
  then $X_H(u,v)$ = number of common neighbours of u and v.
- For simplicity here we restrict to the case of a single root

Strictly balanced rooted graphs

- For some rooted graphs $\mu_H := \mathbf{E}\, X_H(1) \to \lambda \in (0,\infty)$ implies

$$X_H(1) \xrightarrow{\mathrm{d}} \mathrm{Poisson}(\lambda)$$

- Can be characterized structurally: vaguely, H has no 'dense rooted subgraphs', such H we call *strictly balanced*
- Strictly balanced includes cycles and cliques rooted on a vertex
- Does not include trees (other than one edge)
- Example of non-Poisson limit: $H =$ , $np \to c \in (0,\infty)$ then

$$X_H(1) \xrightarrow{\mathrm{d}} \sum_{i=1}^{N} X_i,$$

where $N, X_1, X_2, \ldots$ are i.i.d. $\mathrm{Poisson}(c)$.

Extension count concentration: strictly balanced graphs

Let H be strictly balanced, root v not isolated

Theorem (Spencer 1990)

$$\frac{\max_{x\in[n]} |X_H(x) - \mu_H|}{\mu_H} \xrightarrow{\mathrm{p}} 0, \quad n \to \infty,$$

whenever $\mu_H \gg \log n$.

So maximal deviations are $o(\mu_H)$. But what order precisely?

Theorem (Šileikis & Warnke 2019+)

If root is not isolated in H, *there exist* $c, C, \alpha > 0$ *such that*

$$\mathbf{Pr}\left(\max_{x\in[n]} |X_H(x) - \mu_H| \in [c\sqrt{\mu_H \log n}, C\sqrt{\mu_H \log n}]\right) \to 1$$

provided $1 \ll \mu_H \leq n^{\alpha}$.

Theorem (Šileikis & Warnke 2019+)

If H is strictly balanced and root is not isolated, there exist $c, C, \alpha > 0$ such that

$$\mathbf{Pr}\left(\max_{x\in[n]}|X_H(x) - \mu_H| \in [c\sqrt{\mu_H \log n}, C\sqrt{\mu_H \log n}]\right) \to 1$$

provided $1 \ll \mu_H \le n^\alpha$.

- In this range $\sigma_H^2 := \mathbf{Var} X_H \sim \mu_H$. Moral: *maximal deviation of extension counts is* $\Theta(\sqrt{\log n})$ *standard deviations*
- For larger μ_H can have $\sigma_H^2 \gg \mu_H$.

Conjecture

For H as above exist constants $c, C > 0$ such that for $\mu_H \gg 1, p \le \frac{1}{2}$

$$\mathbf{Pr}\left(\max_{x\in[n]}|X_H(x) - \mu_H| \in [c\sigma_H\sqrt{\log n}, C\sigma_H\sqrt{\log n}]\right) \to 1,$$

What we know for general H

- Recall $\sigma_H := \sqrt{\mathbf{Var} X_H}$
- Suppose p is such that $X_H(1) \geq 1$ whp and $p = 1 - \Omega(1)$

Theorem (Šileikis & Warnke 2019+)

For arbitrary sequences $a_n = o(1)$ and $b_n = n^{\Omega(1)}$

$$\mathbf{Pr}\left(\frac{\max_{x\in[n]} |X_H(x) - \mu_H|}{\sigma_H} \in [a_n, b_n]\right) \to 1$$

- Conjecture: if H is strictly balanced and root not isolated we can take $a_n \asymp b_n \asymp \sqrt{\log n}$
- There exist H such that arbitrary $b_n \to \infty$ suffices
- There exist H such that $b_n \gg \sqrt{\log n}$ is necessary

Problem

For which graphs can we take $b_n \asymp a_n$? In that case, how does it depend on n and p?