Zero-Error Shift-Correcting and Shift-Detecting Codes

Miloš Stojaković

Joint work with Mladen Kovačević and Vincent Y. F. Tan.

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C. E. Shannon, "The Zero Error Capacity of a Noisy Channel", *IRE Trans. Inf. Theory* 2 (3), 1956.

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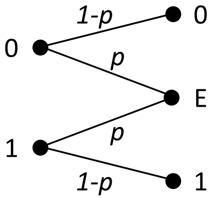

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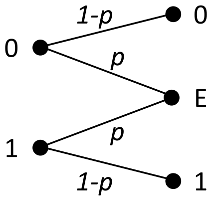

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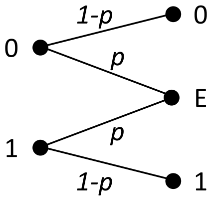

- Observation: No matter which codeword is sent, the sequence EE $\cdots$ E can be received with positive probability.
  - Every two codewords are confusable $\Longrightarrow$ the zero-error capacity of the BEC is equal to zero.

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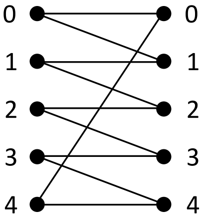
- Another example:



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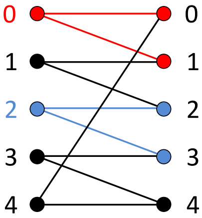

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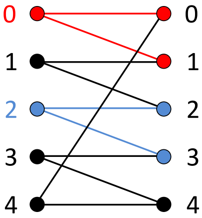

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  - We can transmit one bit per channel use in this way.

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- It turns out that this is the maximal possible rate, i.e., the zero-error capacity of this channel.

L. Lovasz, "On the Shannon Capacity of a Graph", *IEEE Trans. Inf. Theory* 25 (1), 1979.

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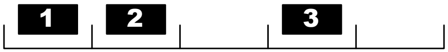

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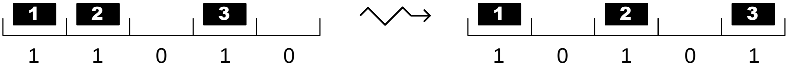

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  - A code is said to be zero-error if no two *sequences* of codewords can produce the same output.

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  - Empty slots at the end of each codeword serve to "catch" the packets that are sent in the preceding slots and are delayed in the channel.
- Restricting to these codes is not a loss in generality (K is a constant):

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathcal{C}(n)| = \lim_{n\to\infty} \frac{1}{n+K} \log |\mathcal{C}(n)|.$$

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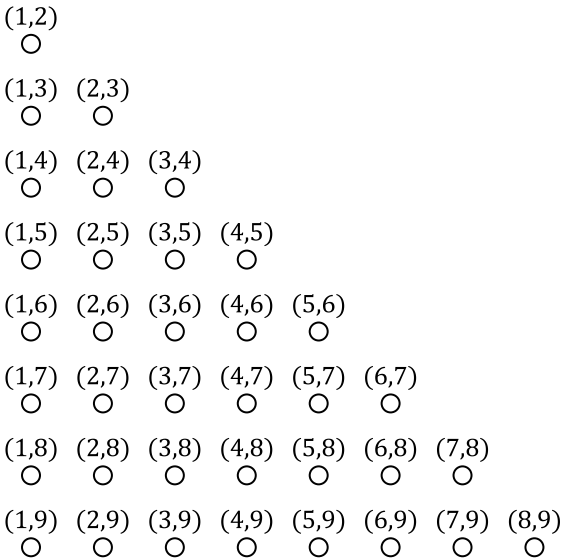
- Example: $n = 9$, $W = 2$, $K = 1$ – Let's try to construct a good code!

Optimal zero-error codes: Geometric approach



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(0,0)							
(0,1)	(1,1)						
(0,2)	(1,2)	(2,2)					
(0,3)	(1,3)	(2,3)	(3,3)				
(0,4)	(1,4)	(2,4)	(3,4)	(4,4)			
(0,5)	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)		
(0,6)	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)	
(0,7)	(1,7)	(2,7)	(3,7)	(4,7)	(5,7)	(6,7)	(7,7)

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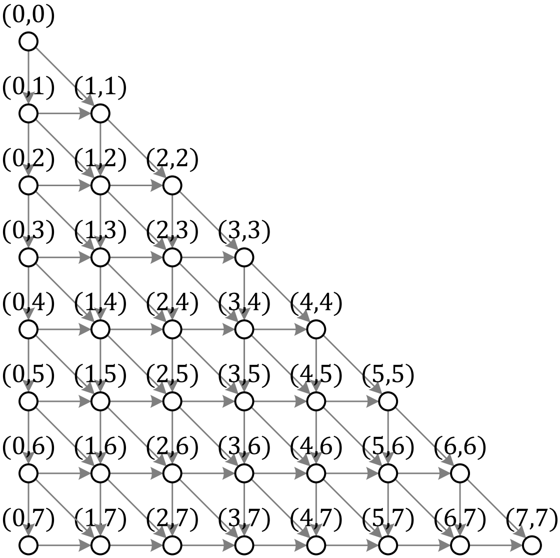

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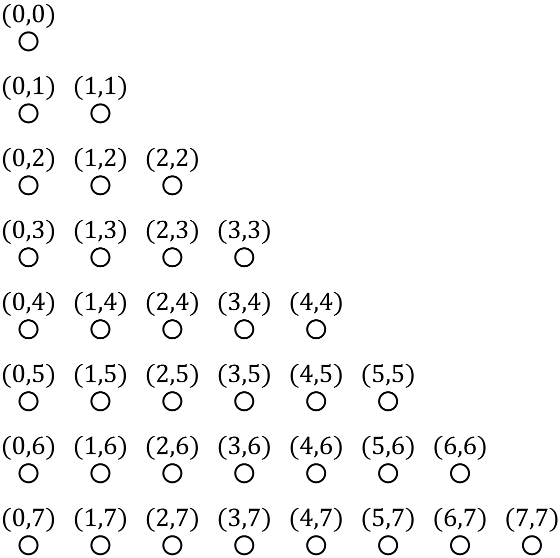

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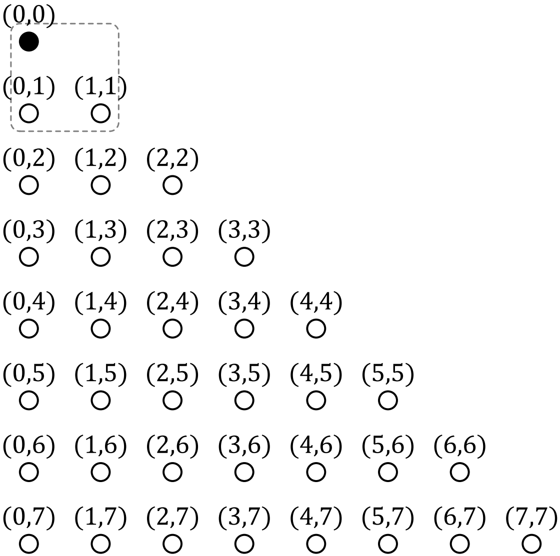

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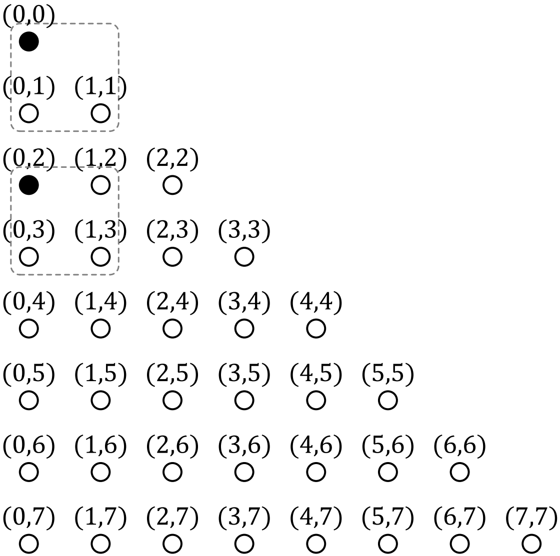

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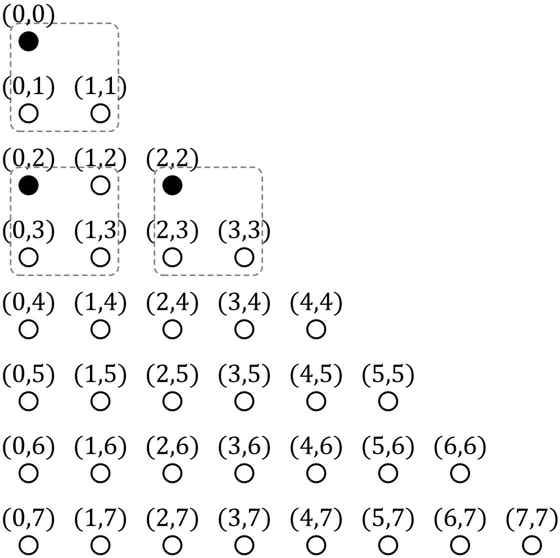

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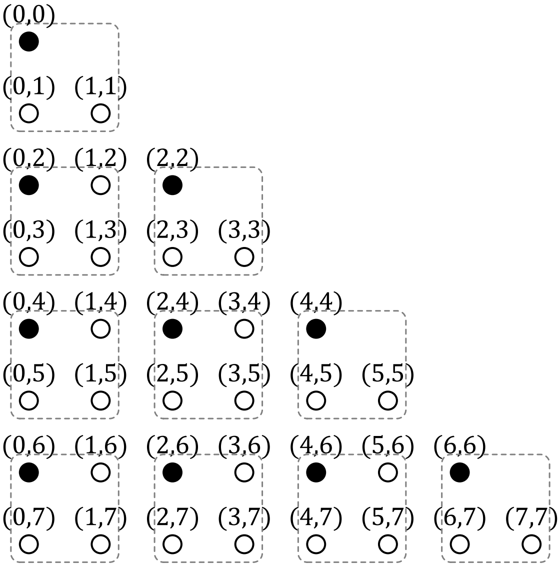

Optimal zero-error codes: Constant-weight case

Thm. Optimal code of length n and weight W is given by

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- The size of the optimal constant-weight code is therefore

$$M(n, W) = \binom{\lfloor \frac{n-W}{K+1} \rfloor + W}{W},$$

$$M(n) = \sum_{W=0}^{n} M(n, W).$$

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Theorem

The zero-error capacity of the Shift Channel with parameter K is equal to $\log r$, where r is the unique positive real root of the polynomial $x^{K+1} - x^K - 1$.

Proof:

- Turns out $M(n)$ can also be described recursively

$$M(n) = M(n-1) + M(n-K-1),$$

with $M(n) = n + 1$ for $n \leq K$.

- This implies that

$$M(n) = \sum_{k=0}^{K} a_k r_k^n,$$

where r_k are the roots of the polynomial $x^{K+1} - x^K - 1$, and a_k are (complex) constants.

- Therefore, $M(n) \sim ar^n$, where r is the largest of these roots (which is the unique positive real root). $\square$

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$$C_0(\omega) = \lim_{n\to\infty} \frac{1}{n} \log M(n, \omega n) = \frac{\omega K + 1}{K+1} \mathcal{H}\left(\frac{\omega(K+1)}{\omega K + 1}\right).$$

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- As there are linearly many different weights, the zero-error capacity can be achieved with constant-weight codes, so

$$C_0 = \max_{\omega\in[0,1]} C_0(\omega) = \frac{\omega^* K + 1}{K+1} \mathcal{H}\left(\frac{\omega^*(K+1)}{\omega^* K + 1} \right),$$

for some ω^*.

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- We can estimate this function further:

$$\frac{1}{n}\log M(n,\omega^* n) = C_0 - \frac{1}{2n}\log n + \mathcal{O}\left(\frac{1}{n}\right).$$

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- Note: Even though the capacity can be achieved with constant-weight codes, their performance is worse at finite blocklengths.
  - ...quantified by the second-order term $-\frac{1}{2n} \log n$

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  - Equivalent to shifts belonging to $\{0, \ldots, K_1 + K_2\}$
- Continuous-time channel with emissions separated by at least τ seconds, and with the maximum delay of T seconds
  - The capacity equals $\frac{1}{\tau} \log r$, where r is the unique positive root of the polynomial $x^{T/\tau} - x^{T/\tau - 1} - 1$

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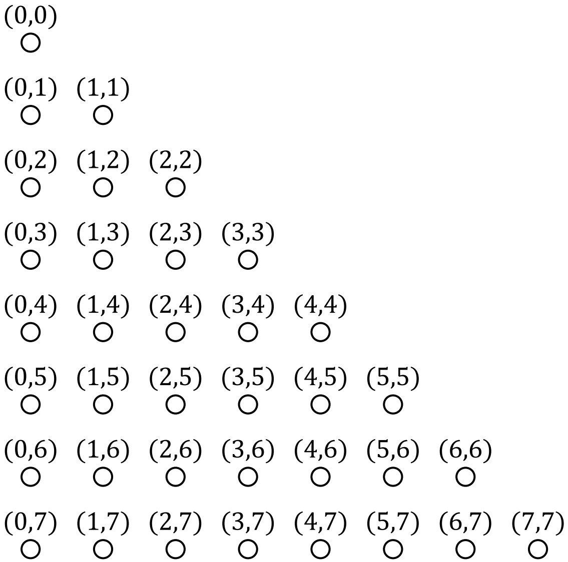
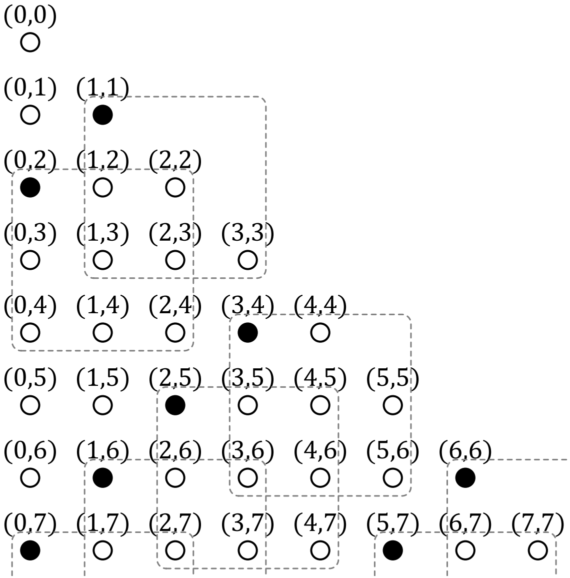
- *Zero-error-detection capacity* of a channel is the largest rate achievable (asymptotically) with zero-error-detecting codes.

Zero-error-detecting codes: Example ($K = 2$)



Zero-error-detecting codes: Example ($K = 2$)



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  - This code is a subcode of $\mathcal{C}(n, W)$ obtained as its intersection with the hyperplanes $\sum_{i=1}^{W} x_i = a \pmod{W K_2 + 1}$

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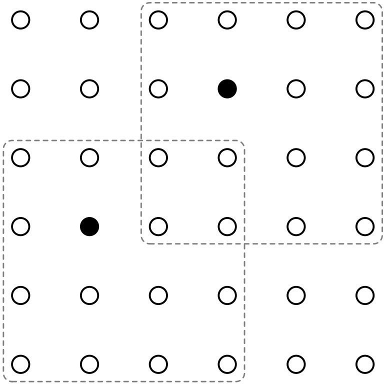

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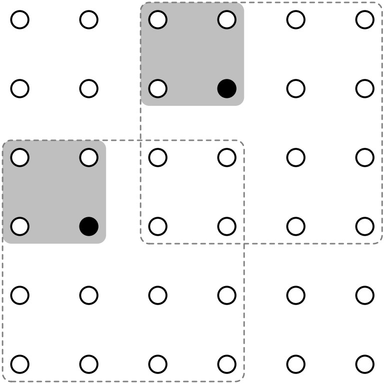

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The zero-error-detection capacity of the Shift Channel with parameters K_1, K_2, *is equal to* $\log r$, *where* r *is the unique positive real root of the polynomial* $x^{\min\{K_1,K_2\}+1} - x^{\min\{K_1,K_2\}} - 1$.

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- ...which is the same as the zero-error-correction capacity of the Shift Channel with parameters 0, $\min\{K_1, K_2\}$.

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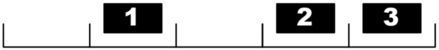

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- Etc.

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