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Intersection problem for linear sets in the projective line

Let $\Lambda = \mathrm{PG}(V, \mathbb{F}_{q^n}) = \mathrm{PG}(r-1, q^n)$, where V is a vector space of dimension r over $\mathbb{F}_{q^n}$. A point set L of Λ is said to be an $\mathbb{F}_q$-*linear set* of Λ of rank k if L is defined by the non-zero vectors of a k-dimensional $\mathbb{F}_q$-vector subspace U of V, i.e.

$$L = L_U = \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\}.$$

Consider two $\mathbb{F}_q$-linear sets L_1 and L_2 in Λ. Clearly, $L_1 \cap L_2$ is still an $\mathbb{F}_q$-linear set of Λ, whenever $L_1 \cap L_2$ is non-empty. Hence, the **intersection problem** can be formulated as follows:

1. do L_1 and L_2 meet in at least one point?
2. if $L_1 \cap L_2 \neq \emptyset$, what is the size of $L_1 \cap L_2$?

An answer to these questions turn out to be difficult in general. The intersection problem for linear sets has been investigated in the following cases:

- two subspaces;
- two subgeometries, see [1];
- one $\mathbb{F}_q$-linear set with one $\mathbb{F}_q$-subline in $\mathrm{PG}(1, q^n)$, see [3, 5];
- two scattered linear sets of rank $n+1$ in $\mathrm{PG}(2, q^n)$, [2];
- one scattered linear set of rank $3n$ with either one line or one plane in $\mathrm{PG}(2n-1, q^3)$, see [4];
- two clubs, see [6].

The intersection problem for linear sets appears in several contexts, such as blocking sets, KM-arcs, PN-functions, irreducible polynomials and semifields.

In this talk we will investigate the intersection problem between two linear sets in the projective line over a finite field. In particular, we analyze the intersection between two clubs with eventually different maximum fields of linearity. Also, we analyze the intersection between the linear set defined as follows

$$\{\langle(x, \alpha x^{q^k} + \beta x)\rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\}$$

and other linear sets having the same rank; this family contains the linear set of pseudoregulus type. The strategy relies on the study of certain algebraic curves whose rational points describe the intersection of the two linear sets. Among other geometric and algebraic tools, function field theory and the Hasse-Weil bound play a crucial role.

This is joint work with Giovanni Zini.

References

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