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Stability of weak saturation number for bipartite graphs

Given two graphs G and F, a spanning subgraph $H \subset G$ is called a *weakly F-saturated subgraph of G*, if it contains no subgraph isomorphic to F and there exists a sequence of graphs $H = H_0 \subset H_1 \subset \ldots \subset H_m = G$ such that, for every $i \in \{1, \ldots, m\}$, H_i is obtained from H_{i-1} by adding an edge that belongs to a copy of F in H_i. The minimum number of edges in a weakly F-saturated subgraph of G is called *weak F-saturation number of G* and denoted by $\mathrm{wsat}(G, F)$.

We consider complete bipartite $F = K_{s,t}$, $s \leq t$. For arbitrary s and t, the exact value of $\mathrm{wsat}(K_n, K_{s,t})$ is not known. However, it is easy to show that $\mathrm{wsat}(K_n, K_{1,t}) = \binom{t}{2}$. Recently [1], it was proven that

$$\mathrm{wsat}(K_n, K_{t,t}) = (t-1)(n+1-t/2)$$

for $n \geq 3t-3$ and

$$\mathrm{wsat}(K_n, K_{t-1,t}) = (t-2)(n+1-(t-1)/2)+1$$

for $n \geq 3t-6$. We have proven that, in all the above cases, the weak saturation number is stable, i.e. with high probability $\mathrm{wsat}(G(n,p), K_{s,t}) = \mathrm{wsat}(K_n, K_{s,t})$, where $G(n,p)$ is the binomial random graph with constant edge probability p. Moreover, we have shown that there exists $C > 0$ such that, for $s = 1$, the stability property holds true for every

$$p \geq C(n[\ln n]^{t-2})^{-1/(t-1)}.$$

This result is sharp: there exists $c > 0$ such that, for $p \leq c(n[\ln n]^{t-2})^{-1/(t-1)}$, with high probability $\mathrm{wsat}(G(n,p), K_{1,t})$ does not equal to $\mathrm{wsat}(K_n, K_{1,t})$.

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References

[1] G. Kronenberg, T. Martins, and N. Morrison, *Weak saturation numbers of complete bipartite graphs in the clique*, arXiv:2004.01289.