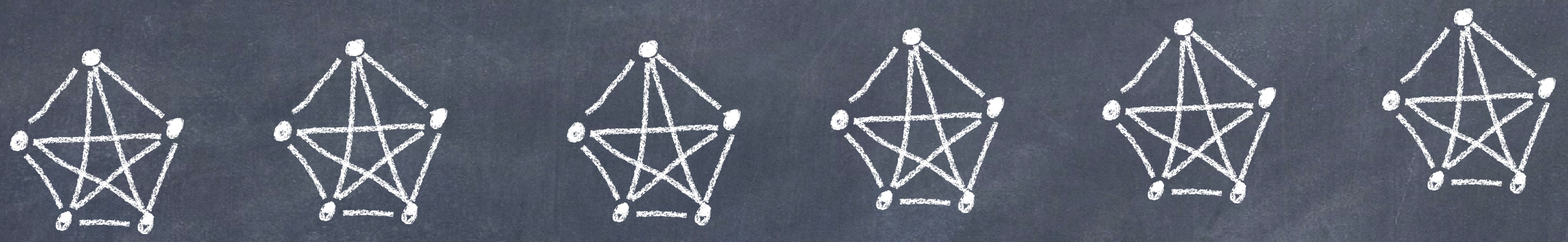
Random Graph Intuition in Maker-Breaker Games

— The Clique-factor game —



Anita Liebenau & Rajko Nenadov

8th Polish Combinatorial Conference 2020

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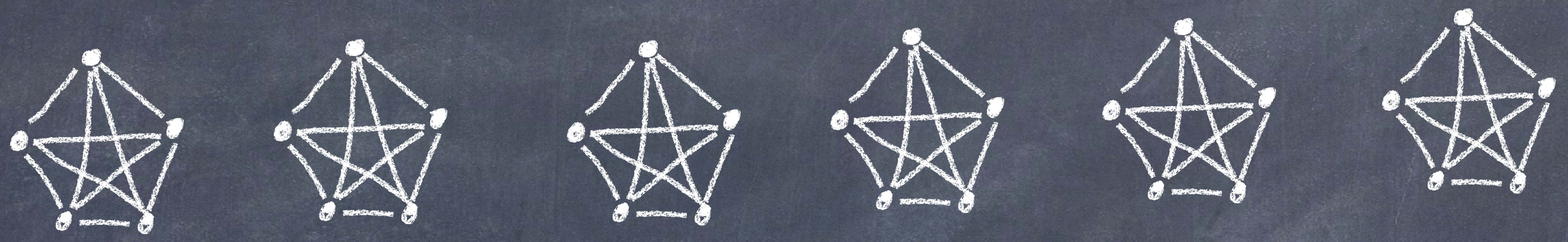

$\frac{n}{r}$ vertex-disjoint copies of K_r

- spanning
- contain 
- r constant, n very large

Maker-Breaker games — the rules

$(X, \mathcal{F})$

in round i: Maker claims 1 element of X
then Breaker claims 1 element of X

Maker wins if Maker occupies all elements of some $F \in \mathcal{F}$

Otherwise Breaker wins

Maker-Breaker games — the rules

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in round i: **Maker** claims 1 element of X
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Examples: $X = E(K_n)$

$\mathcal{F} = \{\text{all copies of } \triangle\}$, $\mathcal{F} = \{\text{all perfect matchings}\}$

$\mathcal{F} = \{\text{all spanning trees}\}$, $\mathcal{F} = \{\text{all Hamilton cycles}\}$

Example : △-game

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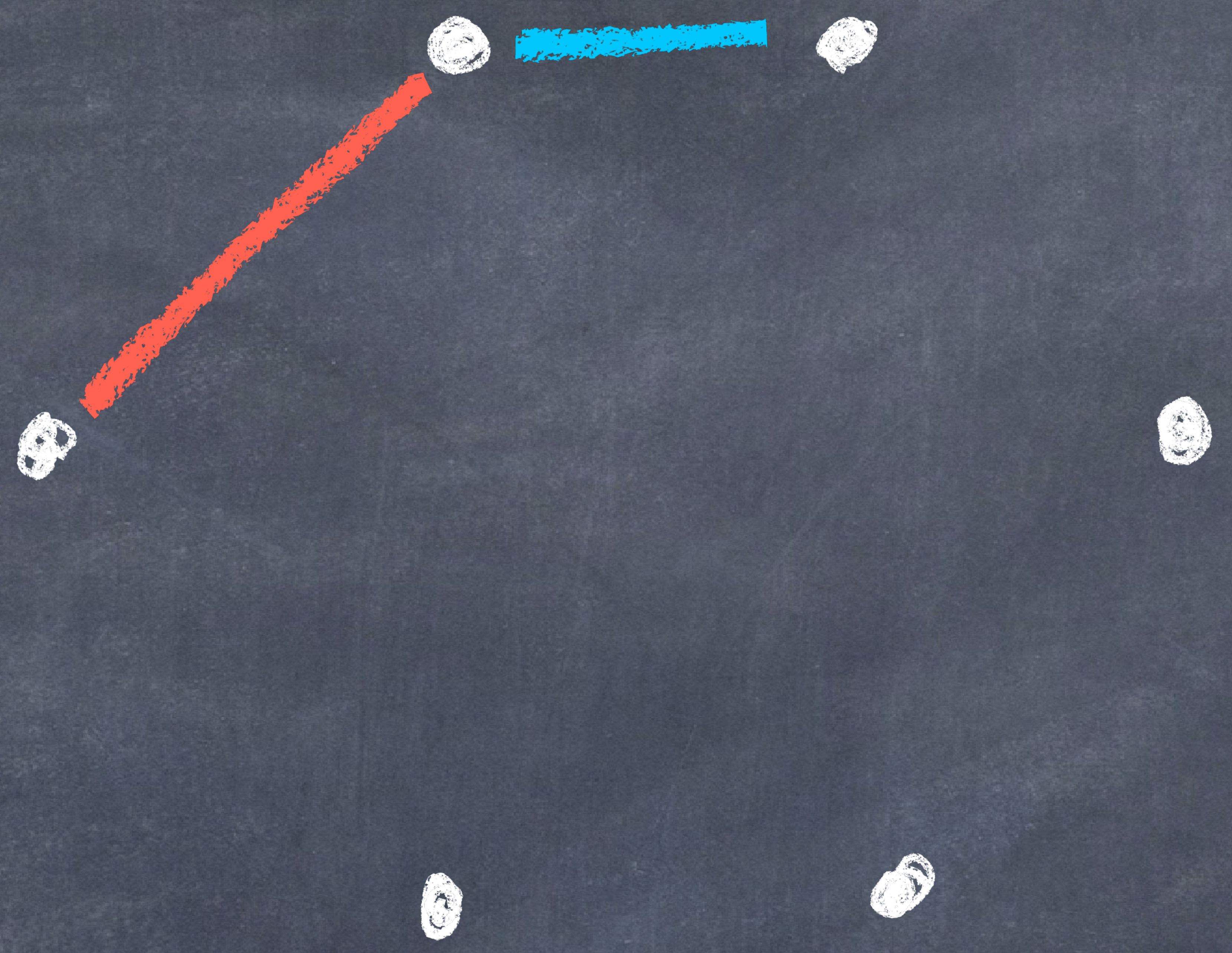

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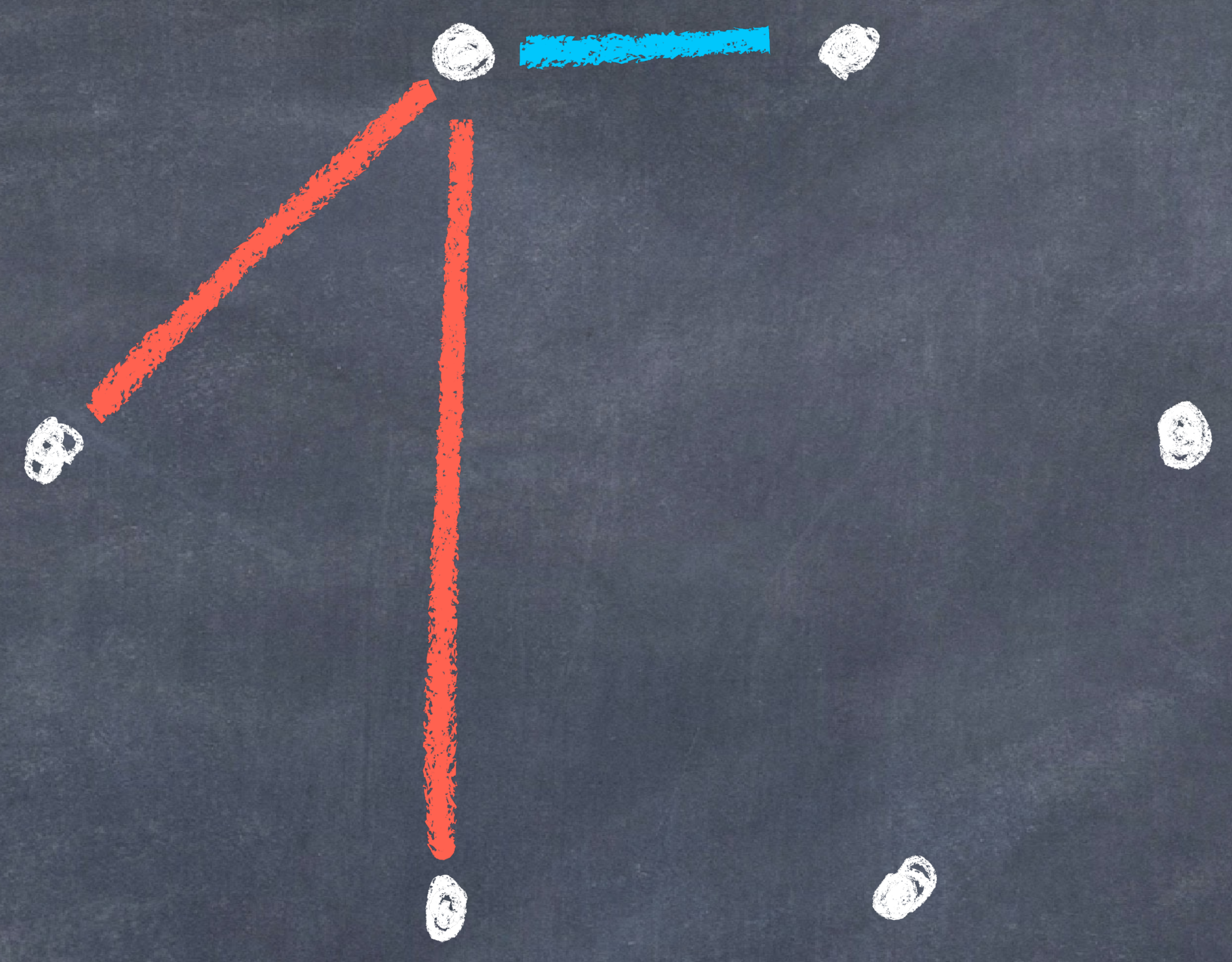

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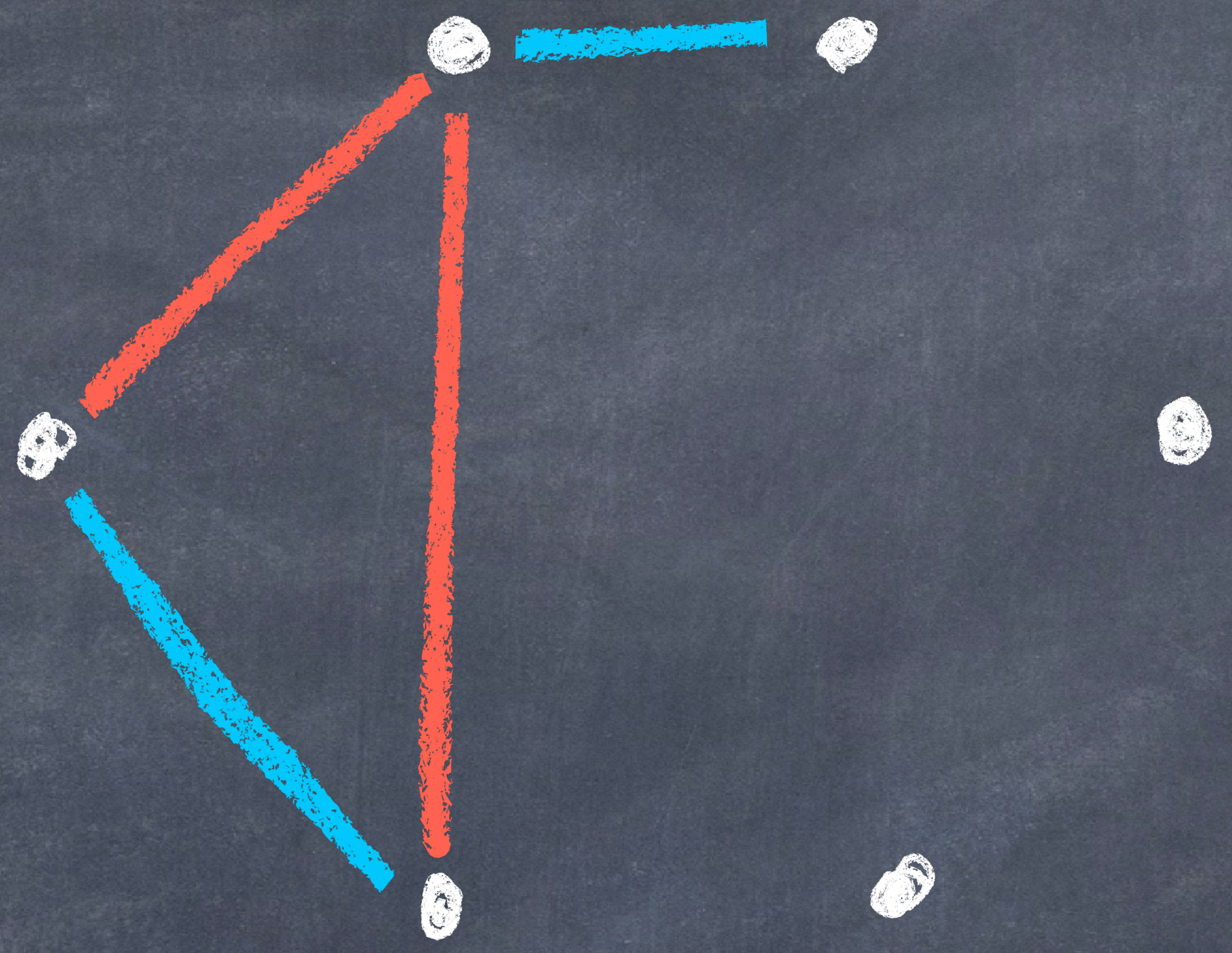

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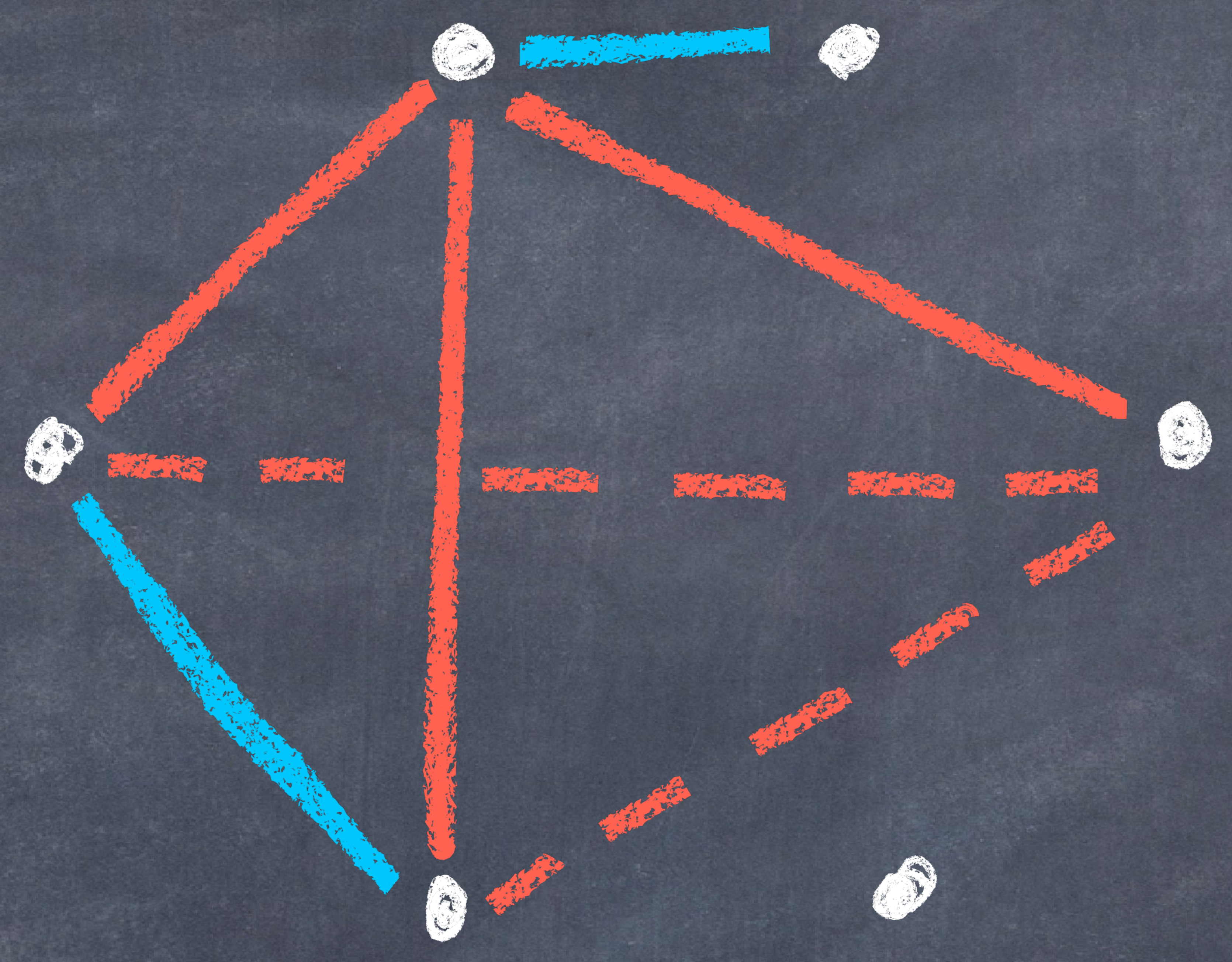
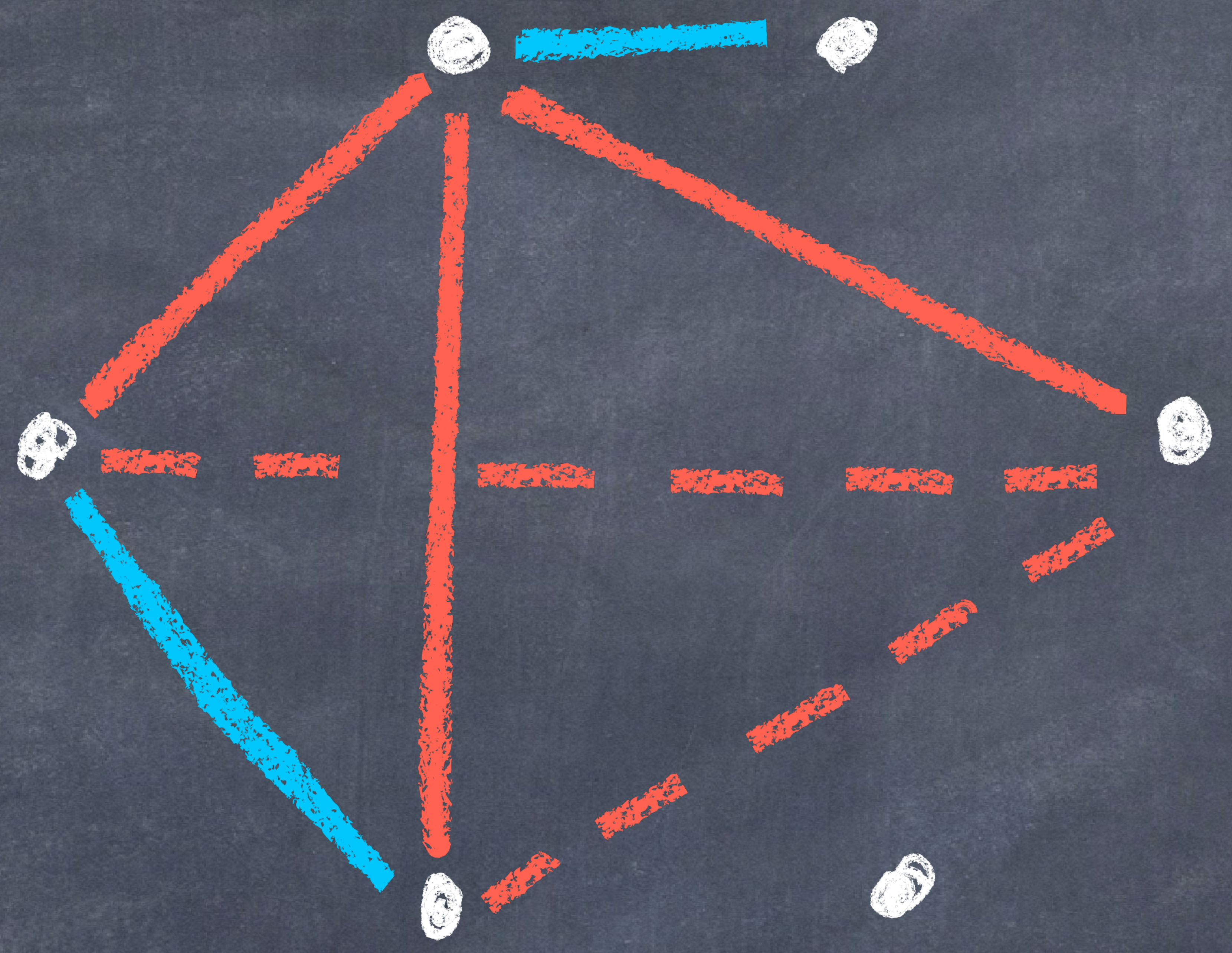


Example : $\triangle$ -game



→ easy win for Maker

→ similarly for perfect matching (PM), Hamiltonicity (HAM) & Connectivity

→ allow Breaker to claim $b = b(n) > 1$ edges per round

Biased Maker-Breaker games on K_n

- b-biased MB games on $E(K_n)$
  - $X = E(K_n)$
  - Breaker claims up to b elements in 1 round

- threshold bias $b^* = b^*(\mathcal{F}, n)$

  smallest b s.t. Breaker wins b-biased game
  ($\mathcal{F} \neq \emptyset$ & $|A| \geq 2 \ \forall A \in \mathcal{F}$)

Threshold biases for some natural games

Connectivity game	$(1+o(1))\frac{n}{\ln n}$	[Chvátal & Erdős '78, Beck '82, Gebauer & Szabó '09]
Perfect Matching game	$(1+o(1))\frac{n}{\ln n}$	[C&E '78, Beck '85, Bollobás & Papaioannou '82, Krivelevich & Szabó '08, Krivelevich '11]
Hamiltonicity game	$(1+o(1))\frac{n}{\ln n}$	
Δ-game	$\Theta(\sqrt{n})$	[C&E '78, Balogh & Samotij '11, Glazik & Srivastav '18+]

Random Game

Maker → RandomMaker :

picks 1 edge u.a.r. from unclaimed edges

Breaker → RandomBreaker :

picks b edges u.a.r. from unclaimed edges

n vertices

Maker's graph $\sim G(n, m)$

where $m = \frac{1}{b+1}\binom{n}{2}$

Threshold biases for some natural games

	Clever Game		Random Game
Connectivity game	$(1+o(1))\frac{n}{\ln n}$	$=$	$(1+o(1))\frac{n}{\ln n}$
Perfect Matching game	$(1+o(1))\frac{n}{\ln n}$	$=$	$(1+o(1))\frac{n}{\ln n}$
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Random Graph Intuition — Erdős paradism

Bednarska & Łuczak 2000

Theorem : H-game ($v(H) \geq 3$)

$\forall$ such H $\exists\, c, C > 0$ s.t. for all n :

$$c\, n^{1/m_2(H)} \leq b^*(H\text{-game}, n) \leq C \cdot n^{1/m_2(H)}$$

- $m_2(H) = \max \left\{ \frac{e(H')-1}{v(H')-2} : H' \subseteq H,\ v(H') \geq 3 \right\}$

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winning strategy for Breaker

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- winning strategy for Maker ⟶ random strategy
- $G(n,m)$ robustly contains H:

$$\forall F \subseteq G(n,m) \text{ s.t. } |F| \leq \varepsilon \cdot m : H \subseteq G(n,m) \setminus F$$

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- winning strategy for Maker $\longrightarrow$ random strategy
- $G(n,m)$ robustly contains H :

  $\forall F \subseteq G(n,m)$ s.t. $|F| \leq \varepsilon \cdot m$: $H \subseteq G(n,m) \setminus F$
- Problem: if winning structure is spanning

Ferber, Krivelevich, Naves 2015

"Local Resilience"

- For every fixed strategy of Breaker Maker draws a random graph $\Gamma \sim G(n,p)$.

> If $C \cdot \frac{\ln n}{n} \leqslant p \leq c \cdot \frac{1}{b}$ then w.h.p. Γ is such that Maker can claim a subgraph $M \subseteq \Gamma$ s.t. $\delta(M) \geq (1-\varepsilon)\, p \cdot n$

$\longrightarrow M = G(n,p) \setminus B$

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- Example: PM, HAM, Connectivity for $b = C \cdot \frac{\ln n}{n}$

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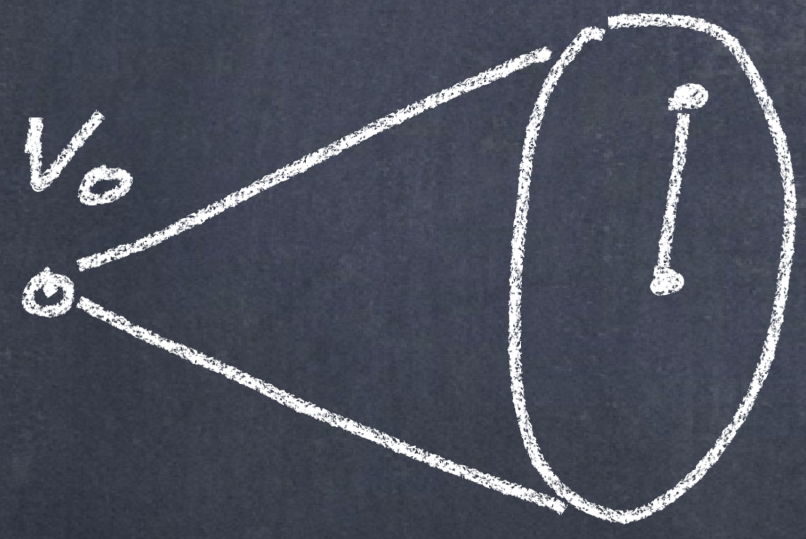


$N(v_0)$ in $G(n,p)$

$\mathbb{E}(\#\Delta\text{'s containing } v_0) \leq p^3 n^2 \ll pn$

[if $p \ll n^{-1/2}$]

→ can destroy all Δ's by deleting $\leq \varepsilon \cdot pn$ edges at every vertex

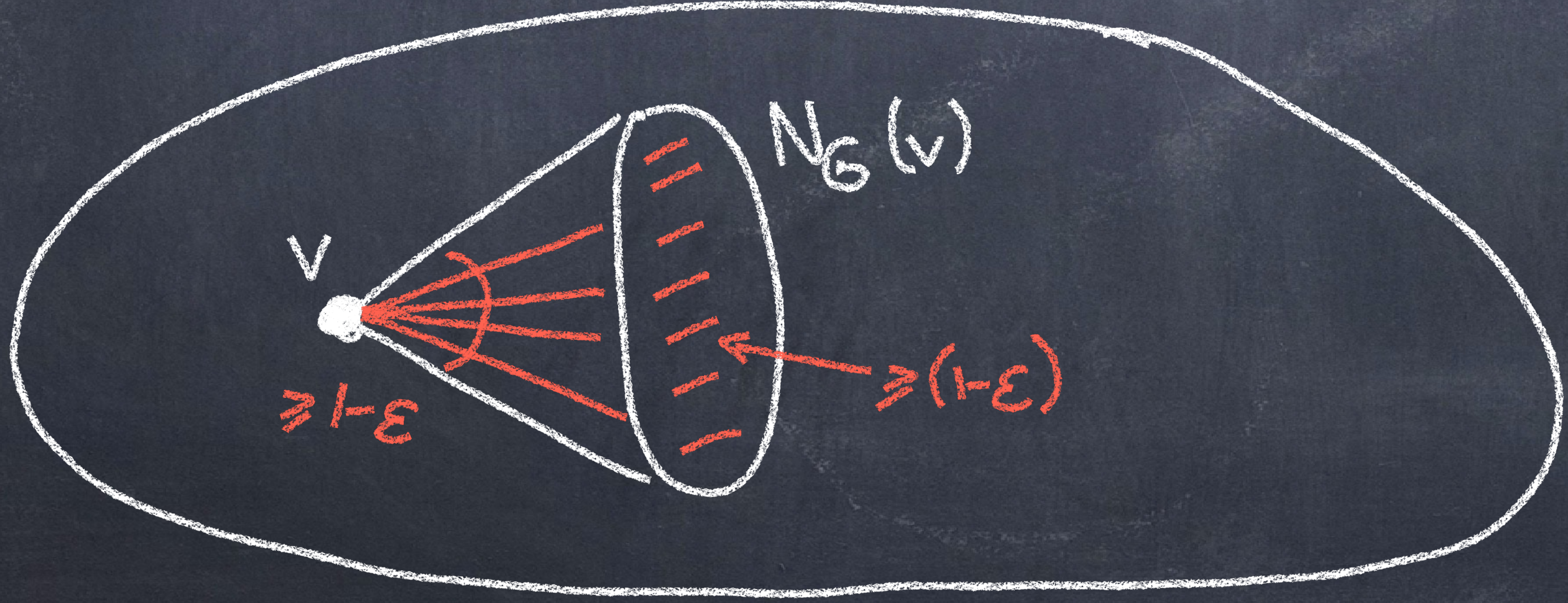
Allen, Böttcher, Kohayakawa, Naves, Person 2017+

Theorem

Let n, ε be given. Let $p \geq 10^8 \varepsilon^{-2} n^{-1/2}$ and $p \leq 10^{-24} \varepsilon^6 b^{-1}$.

In the b-biased game on $E(K_n)$, Maker can claim a subgraph M of $G \sim G(n,p)$ such that for all $v \in V$:

$$d_M(v) \geq (1-\varepsilon)np \quad \text{and} \quad e(M[N_G(v)]) \geq (1-\varepsilon)\frac{p^3 n^2}{2}.$$



$G \sim G(n,p)$

Allen, Böttcher, Kohayakawa, Naves, Person 2017+

Corollary

For all Δ there exists c such that for all n & for all H on n vertices with $\Delta(H) \leq \Delta$: Maker wins the b-biased H-game on K_n if

$$b \leq c \cdot \left(\frac{n}{\log n}\right)^{1/\Delta}$$

Proof:

- $M \subseteq G(n,p)$ s.t. $\delta(M) \geq (1-\varepsilon)np$ & $e(M[N_G(v)]) \geq (1-\varepsilon)\frac{n^2p^3}{2}$
- "Sparse Blow-up lemma" for $G(n,p)$

Allen, Böttcher, Kohayakawa, Naves, Person 2017+

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Remarks: (1) universal result

(2) more general version for "bounded degree & bounded degeneracy"

Allen, Böttcher, Kohayakawa, Naves, Person 2017+

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Example 1: If $b \leq c \cdot \left(\frac{n}{\log n}\right)^{1/2}$ then Maker wins △-factor game.

Remember: Breaker wins △-game for $b \geq 2\sqrt{n}$.

Allen, Böttcher, Kohayakawa, Naves, Person 2017+

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Example 2: If $b \le c \cdot \left(\frac{n}{\log n}\right)^{1/3}$ then Maker wins K_4-factor game.

They prove : Breaker wins K_4-factor game for $b \ge C n^{1/3}$.

Allen, Böttcher, Kohayakawa, Naves, Person 2017+

Corollary

For all Δ there exists c such that for all n & for all H on n vertices with $\Delta(H) \le \Delta$:

Maker wins the b-biased H-game on K_n if

$$b \le c \cdot \left(\frac{n}{\log n}\right)^{1/\Delta}$$

- $b^*(K_{\Delta+1}\text{-factor}, n) = n^{\frac{1}{\Delta} + o(1)}$ for $\Delta = 2, 3$
- "$\frac{1}{\Delta}$ probably not correct for larger Δ"

L. & Nenadov 2020+

Theorem

For all $\Delta \geq 3$ $\exists c$ such that for every $n \in (\Delta+1)\mathbb{Z}$

in the b-biased MB-game on K_n

Maker wins the $K_{\Delta+1}$-factor game if $b \leq c \cdot n^{\frac{2}{\Delta+3}}$

L. & Nenadov 2020+

Theorem

For all $\Delta \geq 3$ $\exists\, c, C$ such that for every $n \in (\Delta+1)\mathbb{Z}$ in the b-biased MB-game on K_n

(a) Maker wins the $K_{\Delta+1}$-factor game if $b \leq c \cdot n^{\frac{2}{\Delta+3}}$

(b) Breaker wins the $K_{\Delta+1}$-factor game if $b \geq C \cdot n^{\frac{2}{\Delta+3}}$

Threshold bias $b^*(\mathcal{F}, n)$

	Clever Game		Random Game
Connectivity, PM, HAM	$(1+o(1))\frac{n}{\ln n}$	$=$	$(1+o(1))\frac{n}{\ln n}$
△- game	$\Theta(\sqrt{n})$	$\neq$	$\Theta(n)$
△- factor game	$n^{\frac{1}{2}+o(1)}$		$n^{\frac{2}{3}+o(1)}$
$K_{\Delta+1}$- factor game	$n^{\frac{2}{\Delta+3}}$		$n^{\frac{2}{\Delta+1}+o(1)}$
	our result		Johansson, Kahn, Vu 2008

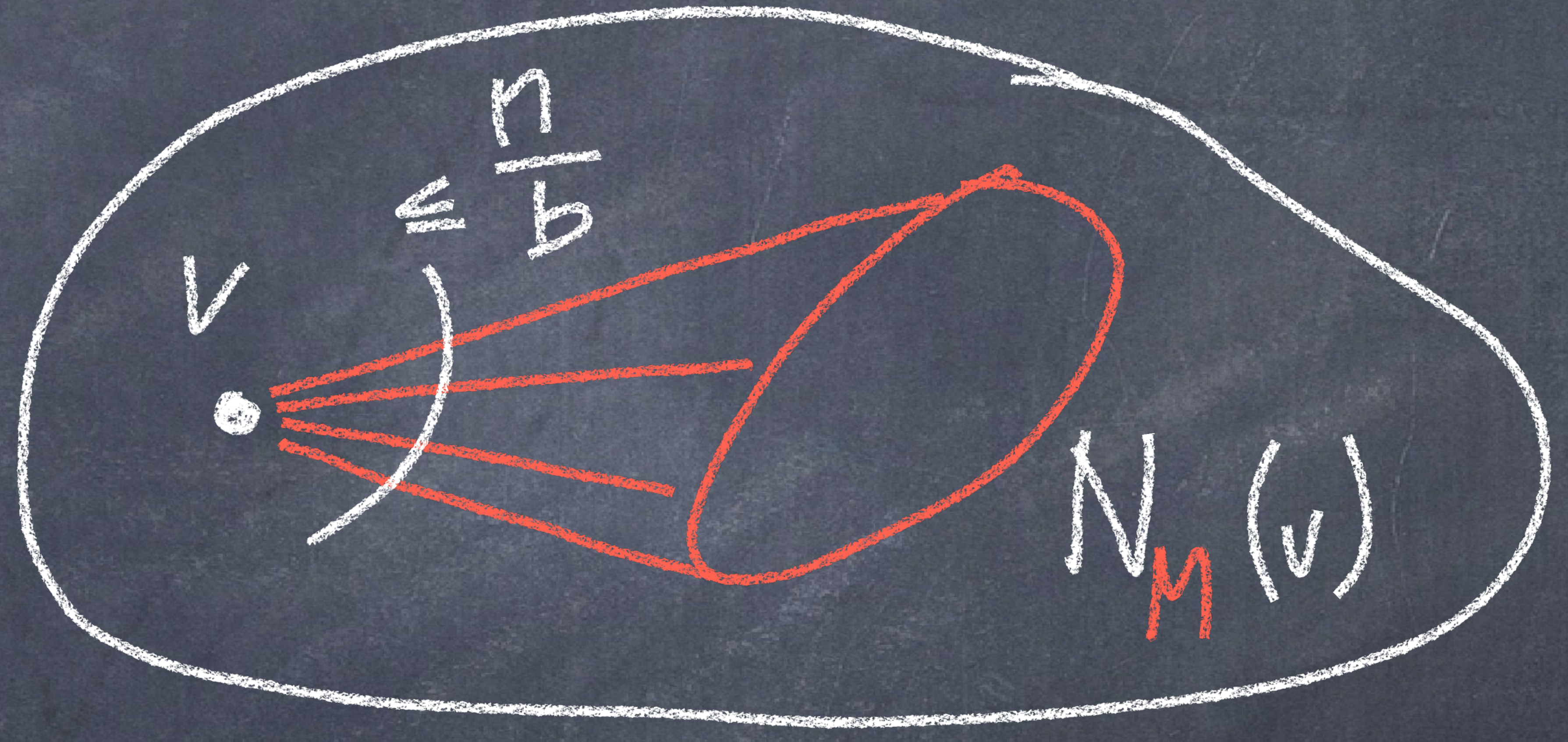
Why $n^{\frac{2}{\Delta+3}}$?

- every v is in a copy of $K_{\Delta+1}$
- For which b can Breaker achieve "$\exists\, v$ that is not in a copy of $K_{\Delta+1}$" ?

Why $n^{\frac{2}{\Delta+3}}$?

- For which b can Breaker achieve "$\exists\, v$ that is not in a copy of $K_{\Delta+1}$" ?

- Fix v
- always play b edges at v
- prevent K_Δ in $N_M(v)$



⟶ use Bednarska-Łuczak

⟶ works if $b \geq C \cdot \left(\frac{n}{b}\right)^{\frac{1}{m_2(K_\Delta)}} \iff b \geq C' n^{\frac{1}{m_2(K_\Delta)+1}} = C' n^{\frac{2}{\Delta+3}}$

Maker's strategy for $b \le cn^{2/(\Delta+3)}$

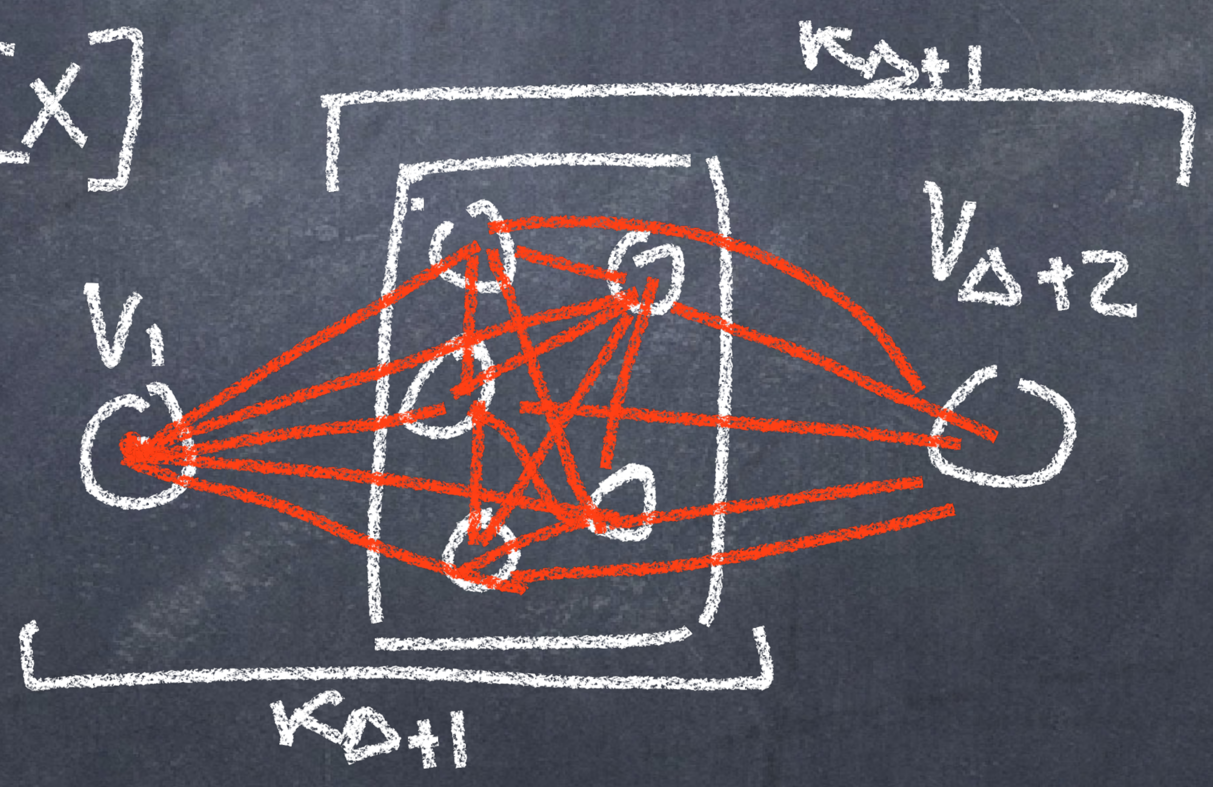
Build a "pseudo-random" graph

(P1) $\delta(M) \ge \frac{np}{2}$ & $\forall X, Y$ $|X| \ge \frac{\log n}{p}$, $|Y| \ge \alpha n$ $\exists v \in X$ s.t.

$$|N_M(v) \cap Y| \ge \frac{|Y|p}{2}$$

(P2) $\forall v \, \forall X \subseteq N(v)$ w/ $|X| \ge \alpha np$: $K_\Delta \subseteq M[X]$

(P3) $\forall V_1, \dots, V_{\Delta+2}$, $|V_i| \ge n^{1-\beta}$ $\exists K^-_{\Delta+2}$:



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Build a "pseudo-random" graph

Apply ABKNP 2017

▷ $M_1 \subseteq G(n,p)$ s.t. $\delta(M_1) \geq (1-\varepsilon)np$ & $e_{M_1}(N_G(v)) \geq (1-\varepsilon)p^3\frac{n^2}{2}$

ε – small constant , $p = c_\varepsilon b^{-1} = Kn^{-\frac{2}{\Delta+3}}$

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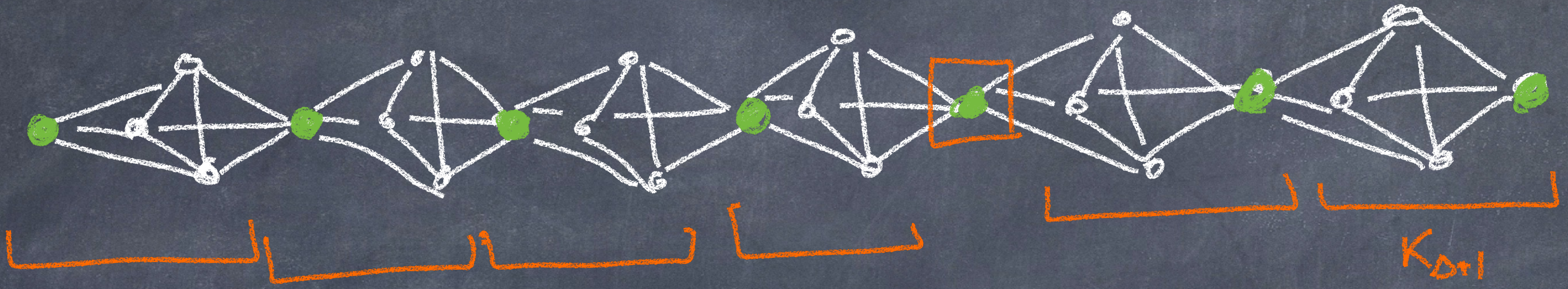
  ε – small constant , $p = c_\varepsilon b^{-1} = Kn^{-\frac{2}{\Delta+3}}$

- $M_2 \subseteq G(n,q)$ s.t. $\delta(M_2) \ge (1-\gamma)nq$ & $e_{M_2}(N_G(v)) \ge (1-\gamma)\frac{q^3 n^2}{2}$

$$n^{-\frac{1-\beta}{m_2(H)}} \ll q \ll n^{-\frac{2}{\Delta+3}}\gamma^6, \quad \gamma = n^{-3\beta}$$

Maker's strategy for $b \leq cn^{2/\Delta+3}$

use (P1), (P2) & (P3) to find chains :



Maker's strategy for $b \leq c n^{2/\Delta+3}$

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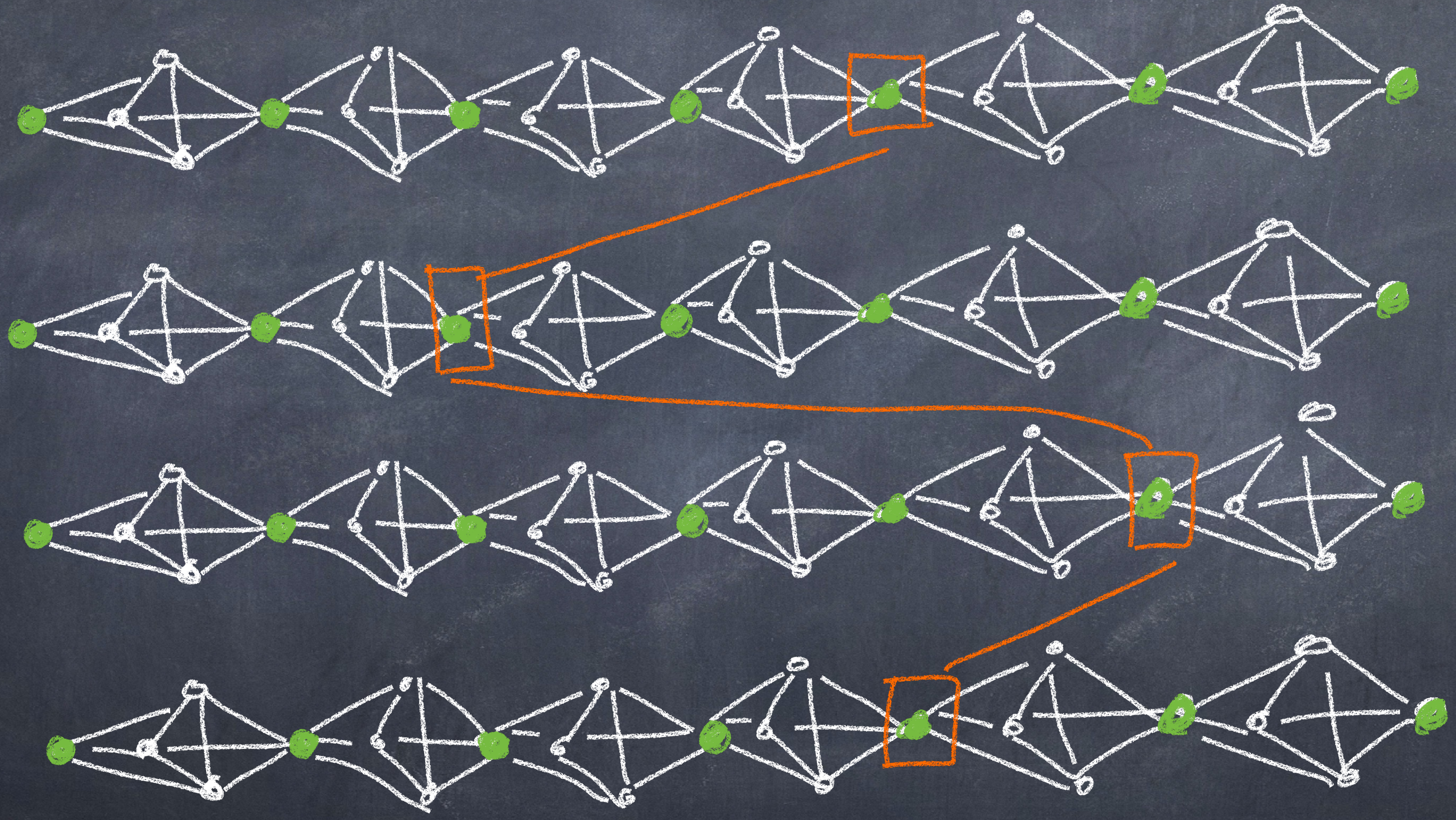

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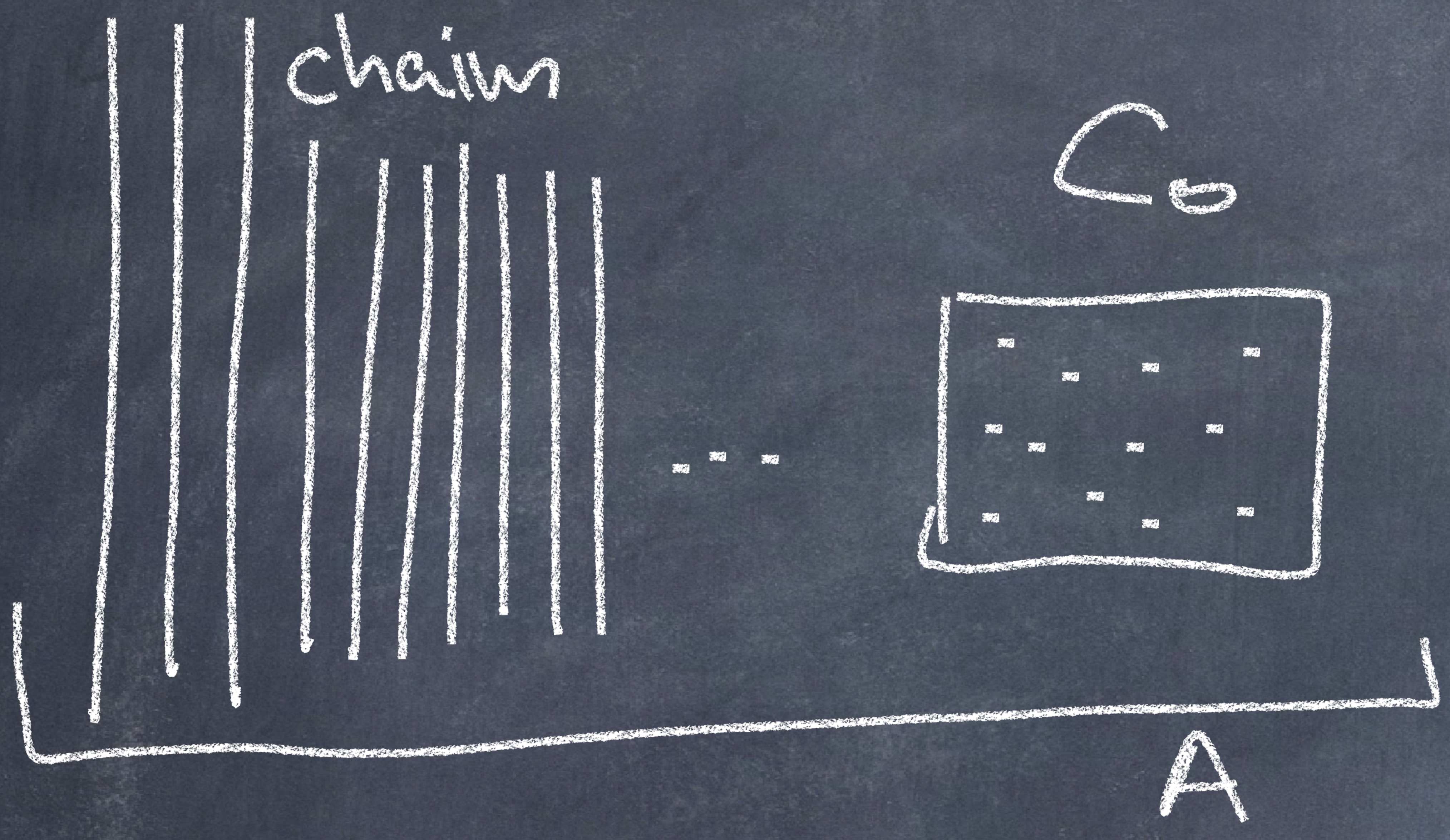

rest of graph

cover $\bar{w}$ $K_{\Delta+1}$'s (somewhat greedily)

using v's from C_0

Absorbing property

$\forall W \subseteq C_0 \ \exists \ K_{\Delta+1}$-factor of $A \setminus W$

Conjecture:

for every H on $\leq n$ vertices with $\Delta(H) \leq \Delta$:

$$b^*(H\text{-game}, n) \geq c \cdot n^{2/\Delta+3}$$

→ $K_{\Delta+1}$-factor is hardest for Maker

Problem:

Find explicit winning strategies for Maker in the H-game, the $K_{\Delta+1}$-factor game, or the Hamiltonicity game.