Automatic Sequences from the perspective of Additive Combinatorics

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8th Polish Combinatorial Conference

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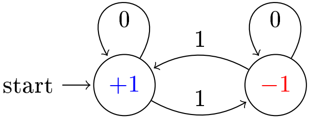

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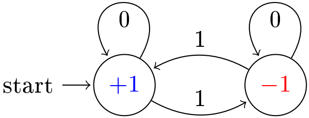

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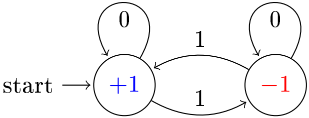

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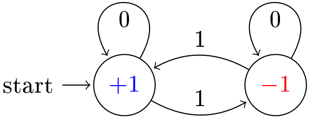

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5. Formal power series: Let $t'(n) = \frac{1-t(n)}{2} \in \{0,1\}$ and $T(z) = \sum_{n=0}^{\infty} t'(n) z^n$. Then

$$z + (1+z)^2 T(z) + (1+z)^3 T(z)^2 = 0 \bmod 2.$$

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2. $\#\{n < N : t(n) = t(n+1)\} \simeq N/3 \neq N/2$. $\longrightarrow$ $t(n) = t(n+1)$ iff $2 \nmid \nu_2(n+1)$
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3. $\sup_{\alpha\in\mathbb{R}} \left| \mathop{\mathbb{E}}_{n<N} t(n)e(n\alpha) \right| = O(N^{-c})$ with $c > 0$. $\longrightarrow$ shorthand: $e(\theta) = e^{2\pi i\theta}$

Gelfond problems

1. The Thue-Morse sequence does not correlate with the primes:

$$\mathop{\mathbb{E}}_{n<N} t(p_n) = O(N^{-c}) \text{ for some } c > 0,$$

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3. The Thue–Morse sequence does not correlate with Piatetski-Shapiro sequences:

$$\mathop{\mathbb{E}}_{n<N} t(\lfloor n^\alpha \rfloor) = O(N^{-c}) \text{ for some } c > 0,$$

where $1 < \alpha < 2$, Spiegelhofer (2020+). Also, $t(n)$ has *level of distribution* 1.

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- More generally, pick a sequence $\epsilon_0, \ldots, \epsilon_{\ell-1} \in \{+1, -1\}$. How many pairs m, n are there with $0 \leq n + im < N$ and $t(n + im) = \epsilon_i$ for all $0 \leq i < \ell$, asymptotically as $N \to \infty$?

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- Even more generally, pick a sequence of affine maps $A_0, A_1, \dots, A_{\ell-1} : \mathbb{Z}^d \to \mathbb{Z}$. How many d-tuples $n_0, n_1, \dots, n_{d-1}$ are there with $0 \leq A_i(n_0, \dots, n_{d-1}) < N$ and $t(A_i(n_0, \dots, n_{d-1})) = \epsilon_i$ for all $0 \leq i < \ell$, asymptotically as $N \to \infty$?

Fourier analysis: first glance

Problem: Let $A \subset [N]$, $\#A = \alpha N$ and $\ell \in \mathbb{N}$. How many ℓ-term arithmetic progressions in A? Is there at least one?

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Motto: A is uniform $\Longleftrightarrow$ $\hat{1}_A(\xi)$ are small for $\xi \neq 0$.

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Suppose that $\left|\hat{1}_A(\xi)\right| < \varepsilon$ for all $\xi \neq 0$. Then

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Corollary: The number of 3-term APs in $\{n \in [N] \, : \, t(n) = +1\}$ is $\simeq N^2/32$.

Fact (Fourier analysis is not enough)

There exist $A \subset [N]$, $\#A = \alpha N$ *such that* $\hat{1}_A(\xi) \simeq 0$ *for* $\xi \neq 0$ *but the number of* 4*-term APs in* A *is not* $\simeq \alpha^4 N^2/6$ *(like for a random set).*

Example: $A = \{n \in [N] : 0 \leq \{n^2\sqrt{2}\} < \alpha\}$. $\longrightarrow$ $\{x\} = x - \lfloor x \rfloor$

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Definition (Gowers norm)

Fix $s \geq 1$. Let $f\colon [N] \to \mathbb{R}$. Then $\|f\|_{U^s[N]} \geq 0$ is defined by:

$$\|f\|_{U^s[N]}^{2^s} = \mathop{\mathbb{E}}_{\mathbf{n}} \prod_{\omega \in \{0,1\}^s} f\left(n_0 + \omega_1 n_1 + \ldots \omega_s n_s\right),$$

where the average is taken over all parallelepipeds in $[N]$, i.e., over all $\mathbf{n} = (n_0, \ldots, n_s) \in \mathbb{Z}^{s+1}$ such that $n_0 + \omega_1 n_1 + \ldots \omega_s n_s \in [N]$ for all $\omega \in \{0,1\}^s$.

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Higher order Fourier analysis: Basic properties

Facts:

1. $\|f\|_{U^s[N]}$ is well-defined for $s \geq 1$, i.e., the average on the RHS is ≥ 0
2. $\|f\|_{U^1[N]} = |\mathbb{E}_n f(n)|$ and $\|f\|_{U^2[N]} \doteq \|\hat{f}\|_{\ell^4}$ $\longrightarrow$ true in $\mathbb{Z}/N\mathbb{Z}$ rather than $[N]$
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Theorem (Generalised von Neumann Theorem)

Fix $s \geq 1$. If $A \subset [N]$, $\#A = \alpha N$ and $\|1_A - \alpha 1_{[N]}\|_{U^s[N]} \leq \varepsilon$, then A contains as many $(s+1)$-term APs as a random set of the same size, up to an error of size ε:

$$\#\{(n,m) \in [N]^2 \;:\; n, n+m, \dots, n+sm \in A\} = \alpha^s N^2/2s + O(\varepsilon N^2).$$

Gowers uniform sequences

Let μ denote the Möbius function

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 \dots p_k \text{ where } p_1, \dots, p_k \text{ are distinct primes,} \\ 0 & \text{if } n \text{ is divisible by a square.} \end{cases}$$

Recall that μ is *multiplicative*, meaning that $\mu(mn) = \mu(m)\mu(n)$ if $\gcd(m, n) = 1$.

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Theorem (Green & Tao (2008+2012))

Fix $s \geq 2$. The Möbius function is Gowers uniform of order s:

$$\|\mu\|_{U^s[N]} \to 0 \textit{ as } N \to \infty.$$

Hence, the primes contain many arithmetic progressions of length $s+1$.
$\longrightarrow$ *Vast over-simplification, quantitative bounds needed, etc.*

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Fix $s \geq 2$. The Möbius function is Gowers uniform of order s:

$$\|\mu\|_{U^s[N]} \to 0 \text{ as } N \to \infty.$$

Hence, the primes contain many arithmetic progressions of length $s + 1$.
$\longrightarrow$ *Vast over-simplification, quantitative bounds needed, etc.*

Theorem (Frantzikinakis & Host (2017))

Let ν be a (bounded) multiplicative function and $s \geq 2$. Then

$$\|\nu\|_{U^s[N]} \to 0 \text{ as } N \to \infty \text{ if and only if } \|\nu\|_{U^2[N]} \to 0 \text{ as } N \to \infty.$$

Higher order Fourier analysis & Thue–Morse

Recall: $t(n) = \begin{cases} +1 \text{ if the sum of binary digits of } n \text{ is even,} \\ -1 \text{ if the sum of binary digits of } n \text{ is odd.} \end{cases}$

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Fix $s \geq 1$ and let $c = c_s$ be as above. Then for any $\epsilon_i \in \{+1, -1\}$, $(0 \leq i \leq s)$

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- Same holds for the Rudin–Shapiro sequence (count appearances of the pattern 11 instead of 1) as well as other pattern-counting sequences.

Higher order Fourier analysis & k-multiplicative sequences

Definition

Fix $k \geq 2$. A sequence $f : \mathbb{N} \to \mathbb{C}$ is *k-multiplicative* if

$$f(n+m) = f(n)f(m) \qquad \text{for all } n, m \text{ s.t. } m < k^i,\ k^i | n \text{ for some } i.$$

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Example

The Thue–Morse sequence $t(n)$ is 2-multiplicative. More generally, let

$$s_k(n) = \text{sum of digits of } n \text{ in base } k.$$

Then $e(\alpha s_k(n))$ is k-multiplicative for any $\alpha \in \mathbb{R}$. $\longrightarrow e(\theta) = e^{2\pi i \theta}$

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Theorem (Fan & K. (2019))

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Finite automata

Finite automata

Some notation: We let k denote the base in which we work. $\longrightarrow$ e.g. $k = 10, k = 2$

- $\Sigma_k = \{0, 1, \ldots, k-1\}$, the set of digits in base k;
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A finite k-automaton consists of:

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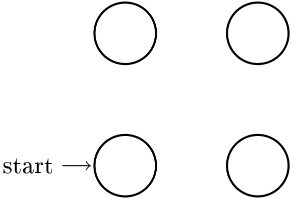

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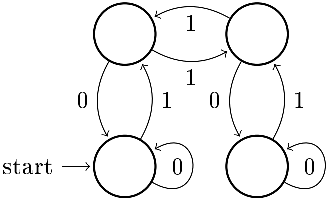

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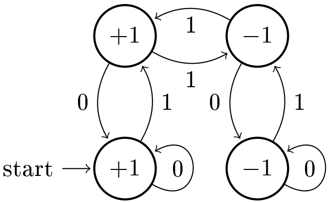

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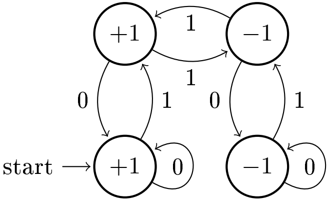

Computing the sequence:

- Extend δ to a map $S \times \Sigma_k^*$ with $\delta(s, uv) = \delta(\delta(s, u), v)$;
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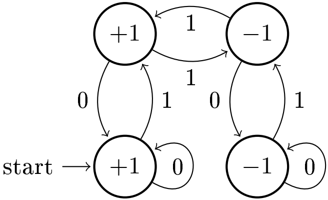

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Motto: Automatic $\Longleftrightarrow$ Computable by a finite device.

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Let $a\colon \mathbb{N} \to \Omega$ be a sequence. Then a is k-automatic if and only if it satisfies any/all of the following equivalent conditions:

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Remark

The many alternative definitions create connections to combinatorics (graph theory, combinatorics on words), computer science, dynamics (symbolic systems), algebra, logic (Büchi arithmetic), etc.

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The following sequences are 2-automatic and not Gowers uniform:

1. periodic sequences like 1 or $(-1)^n$;
2. almost periodic sequences like $\nu_2(n)$ mod 2; $\longrightarrow 2^{\nu_2(n)} || n$
3. slowly varying sequences like $\lfloor \log_2(n) \rfloor$ mod 2. $\longrightarrow$ length of expansion

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Theorem (Byszewski, K. & Müllner (2020+))

Any automatic sequence has a decomposition $a = a_{\text{str}} + a_{\text{uni}}$, where a_{uni} is highly Gowers uniform and a_{str} is a combination of sequences of the above types.

Arithmetic regularity lemma for automatic sequences

Definition: A k-automatic sequence $a(n)$ is *forwards synchronising* if there exists a string of base-k digits $w \in \Sigma_k^*$ such that

$$a([xwy]_k) = a([x'wy]_k)$$

for all strings of base-k digits $x, x', y \in \Sigma_k^*$. Accordingly, $a(n)$ is *backwards synchronising* if there exists $w \in \Sigma_k^*$ such that

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- This is a distant relative of the celebrated Arithmetic Regularity Lemma, which gives a similar decomposition for an arbitrary sequence, albeit with less well-behaved components, Green & Tao, (2010).

Applications and consequences

Corollary

Let a be a k-automatic sequence with $\|a\|_{U^2[N]} \to 0$ as $N \to \infty$. Then for each $s \geq 2$ there exists $c_s > 0$ such that $\|a\|_{U^s[N]} \ll N^{-c_s}$.

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For general sets $A \subset \mathbb{N}$, the corresponding statement is

- true for $\ell = 2, 3, 4$;
- false for all $\ell \geq 5$.

Digression: automatic generalised polynomials

Question: *How to detect lack of Gowers uniformity?*

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Inverse Theorem for Gowers uniformity norms (Green, Tao & Ziegler (2012))

Let $a(n)$ be a bounded sequence. Then the following are (essentially) equivalent:

1. a is not Gowers uniform of all orders;
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- "Extreme case" when investigating uniformity of automatic sequences.
- Alternatively, one can ask how simple a generalised polynomial sequence can be from the point of view of computability, without being trivial.

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- If $A \subset \mathbb{N}$ is 2-automatic then either (i) A has *much additive structure* (contains a "shifted IP set"), or (ii) *looks like* $\{2^i : i \geq 0\}$ ("2-arid").
- If $A \subset \mathbb{N}$ is generalised polynomial and $d(A) = 0$, then A has *very little additive structure* (no IP sets or their shifts), Byszewski & K. (2018).
- If $A \subset \mathbb{N}$ is generalised polynomial and $2^i \in A$ for *many* values of i (central set), then there also are many (syndetic set) values of n such that $2^i n \in A$ for *many* and values of i. K. (2020+).

Digression: automatic generalised polynomials

Question: Are there non-trivial generalised polynomials that are automatic?

- If $\lfloor \alpha n + \beta \rfloor \bmod m$ is automatic then it is periodic, Allouche & Shallit (2003).
- If $a(n)$ is both automatic and a generalised polynomial, then $a =$ [periodic] + [almost everywhere zero], Byszewski & K. (2020).
- If $A \subset \mathbb{N}$ is 2-automatic then either (i) A has *much additive structure* (contains a "shifted IP set"), or (ii) *looks like* $\{2^i : i \geq 0\}$ ("2-arid").
- If $A \subset \mathbb{N}$ is generalised polynomial and $d(A) = 0$, then A has *very little additive structure* (no IP sets or their shifts), Byszewski & K. (2018).
- If $A \subset \mathbb{N}$ is generalised polynomial and $2^i \in A$ for *many* values of i (central set), then there also are many (syndetic set) values of n such that $2^i n \in A$ for *many* and values of i. K. (2020+).

Theorem (Byszewski & K.)

Let $a(n)$ be a sequence and let $k \geq 2$. Then the following are equivalent:

- *$a(n)$ is both a generalised polynomial and a k-automatic sequence;*
- *$a(n)$ is eventually periodic.*

Thank you for your attention!