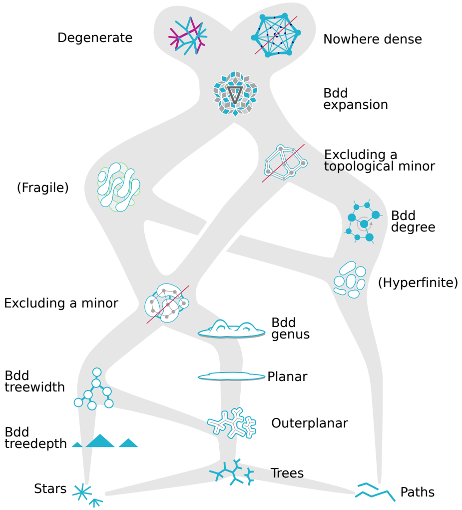
# Graph structure useful for approximation problems

Marcin Wrochna, University of Oxford




*Based on work with Miguel Romero, Standa Živný. Figure based on Felix Reidl's.*

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and hard on some bounded degree graphs.
Where is the boundary?
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We need (much) more expressive problems.

# Constraint Satisfaction Problems (CSPs)

You probably know some of:

Max-2-SAT $(x \vee \neg y) \wedge (y \vee \neg z) \wedge \ldots$

Max-Cut $v \mapsto$ either $\mathtt{L}$ or $\mathtt{R}$, maximize $\#_{uv} : u \in \mathtt{L}, v \in \mathtt{R}$

Max-5-Coloring $v \mapsto$ color, maximize $\#_{uv} : col(u) \neq col(v)$

# Constraint Satisfaction Problems (CSPs)

In general Max-2-CSP the input is:

- graph $G$
- alphabet $\Sigma_v$ for each vertex $v$
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**Theorem (Grohe, Schwentick, Segoufin '01)**

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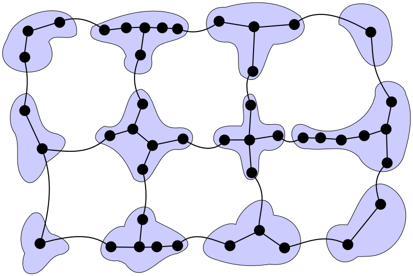
**Grid Minor Theorem (Robertson & Seymour '86)**

$G$ has large tw $\quad\Leftrightarrow\quad$ $G$ contains a large grid as a minor



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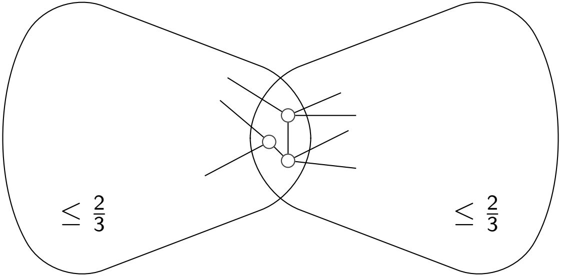
Theorem (Dvořák, Norin '18)

$G$ has bounded treewidth iff
every subgraph has a bounded balanced separation.



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**Theorem (Grohe '07)**

Hom($\mathcal{G}$, $*$) is P $\quad\Leftrightarrow\quad$ $\mathcal{G}$ has bounded treewidth up to hom. eq.

(for example, bipartite graphs are easy).

# Fixed-parameter-tractability

Given $G$ and $k$, decide of $\mathrm{opt}(G) \leq k$ in time $f(k) \cdot |G|^c$.

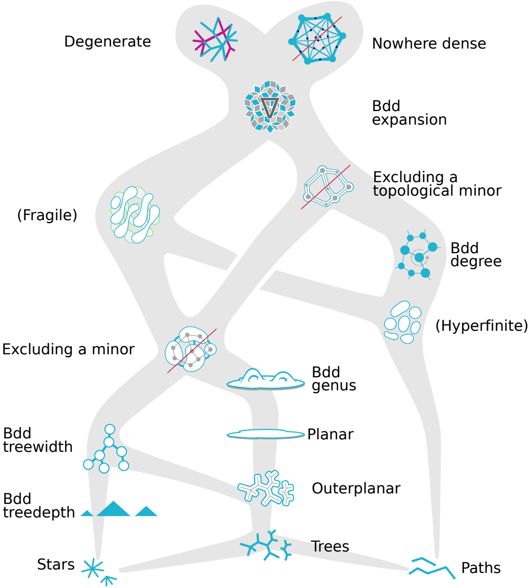
Given $G$ and $\phi$, decide $G \models \phi$ in time $f(|\phi|) \cdot |G|^c$.

For properties expressible by $\mathrm{MSO}_2$ formulas,
this is possible in $\mathcal{G} \quad \Leftrightarrow \quad \mathcal{G}$ has bounded treewidth.

For properties expressible by FO formulas,
this is possible in $\mathcal{G} \quad \Leftrightarrow \quad \mathcal{G}$ is *nowhere-dense*.

Sparse graph theory by: Gaifman, Courcelle, Arnborg, Lagergreen, Seese, Adler, Downey, Fellows, Frick, Grohe, Flum, Nešetřil, Ossona de Mendez, Dawar, Dvořák, Král, Thomas, Kreutzer, Siebertz, ...

# Notions of sparsity



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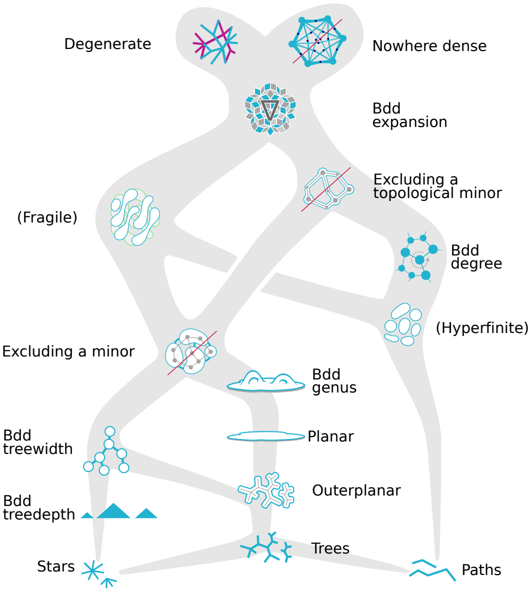


$\mathcal{G}$ is degenerate

$\Updownarrow$

graphs in $\mathcal{G}$ and their subgraphs have bounded average degree

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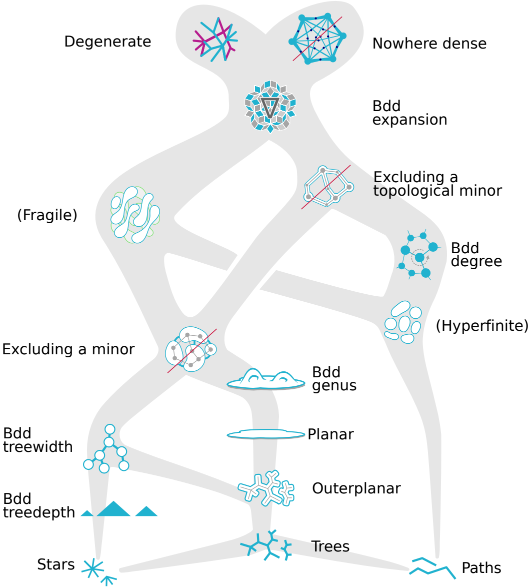


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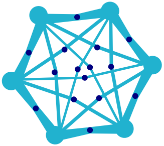

1-subdivision of any graph has average degree $\leq 4$

So $\mathcal{G} = \{$1-subdivisions$\}$ is degenerate, but hard.

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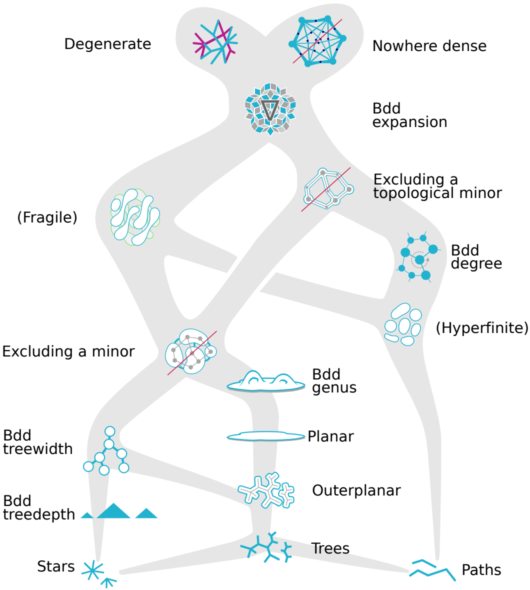


$\mathcal{G}$ is nowhere-dense

$\Updownarrow$

$\forall_r$ not all graphs have an $\leq r$-subdivision in $\mathcal{G}$

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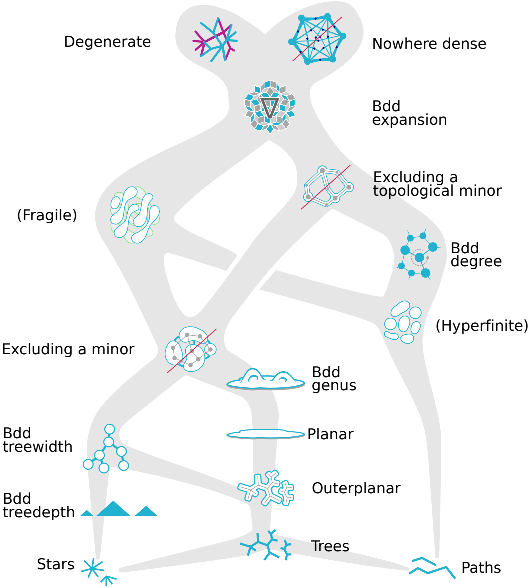


**Conjecture**

Max-CSP($\mathcal{G}$) has a PTAS

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$\mathcal{G}$ is fragile

# Lipton-Tarjan

If graph has $o(n)$ balanced separators, divide & conquer.

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Works for some problems, not so much for general MaxCSP.

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**General intuition:** either $o(n)$ separators or expanders?

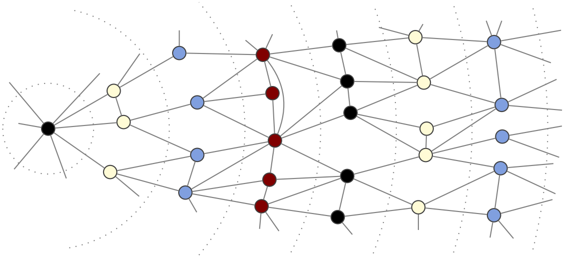
One actually needs $O(\frac{n}{\log n})$, up to $\log \log n$ factors.
(Moshkovitz, Shapira '15)

# Baker's technique (FOCS'83)

In a planar graph, many problems have a PTAS as follows:

- color in layers by distance mod $\lceil \frac{1}{\varepsilon} \rceil$
- some color hits OPT at most $\varepsilon$ times
- try each color $c$: remove it;

  remaining graph has bounded treewidth.



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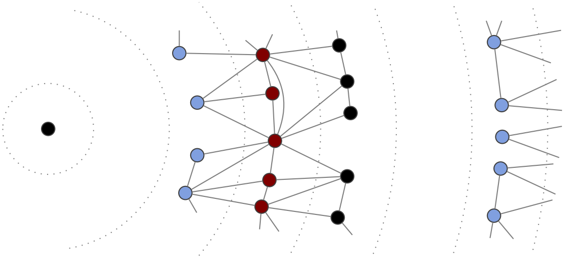

What other graph classes can be partitioned into layers of [. . . ]
bounded treewidth?

– graphs embeddable in a surface $S$, $H$-minor-free graphs
– graph embeddable in plane with $\leq c$ crossings per edge
– some geometric intersection graphs

## Dvořák '16

$\mathcal{G}$ is *fractionally*-tw-**fragile** if for every $\varepsilon > 0$,
every $G \in \mathcal{G}$ has **a distribution of sets** $X \subseteq V(G)$ **such that:**

- removing $X$ reduces treewidth to $\leq k(\varepsilon)$ and
- each vertex is removed with probability $\leq \varepsilon$.

– more general: 3-dimensional grids, . . .
– simpler proofs
– more useful than just Lipton-Tarjan
– robust (treewidth $\leftrightarrow$ treedepth, vertex $\leftrightarrow$ edge, . . . )

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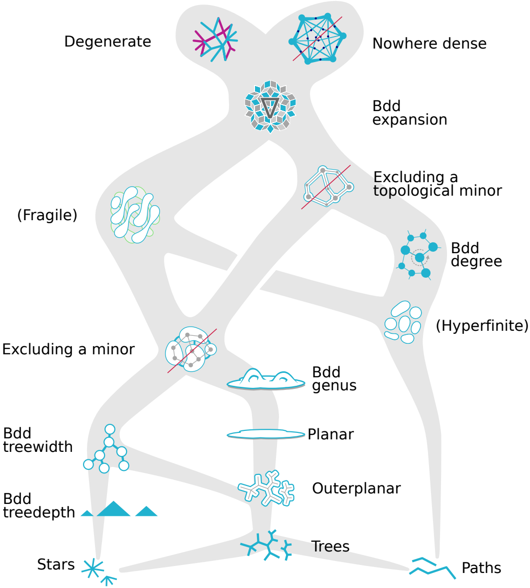


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$\Uparrow$ **Theorem** (Dvořák,R<u>W</u>Ž)

Algorithm: simply run Sherali-Adams LP relaxation

$\Downarrow$ **Questions, questions...**

- non-degenerate $\Rightarrow$ hard?
- high girth 3-regular $\Rightarrow$ hard?
- contains expanders $\Rightarrow$ hard?

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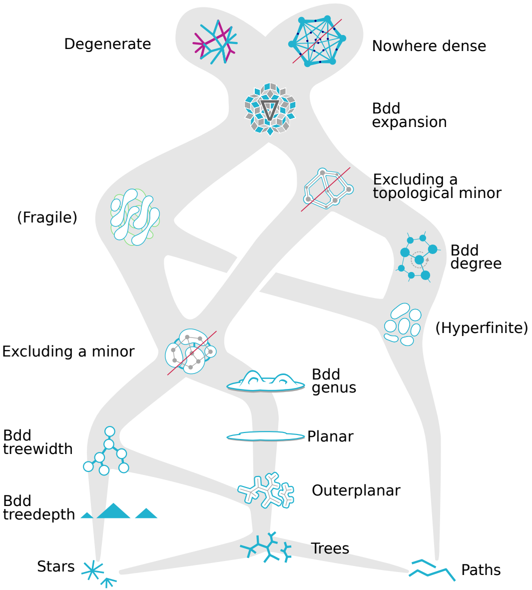


**Theorem** (RW̲Ž)

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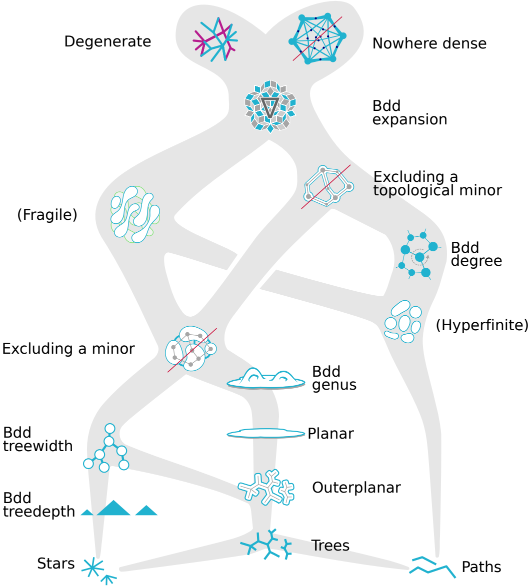


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In group theory:
**Corollary** (Elek)
*uniformly locally amenable*
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*property A*

In property testing:
**Question**
Can results on property testing hyperfinite graphs be extended to unbdd degree?

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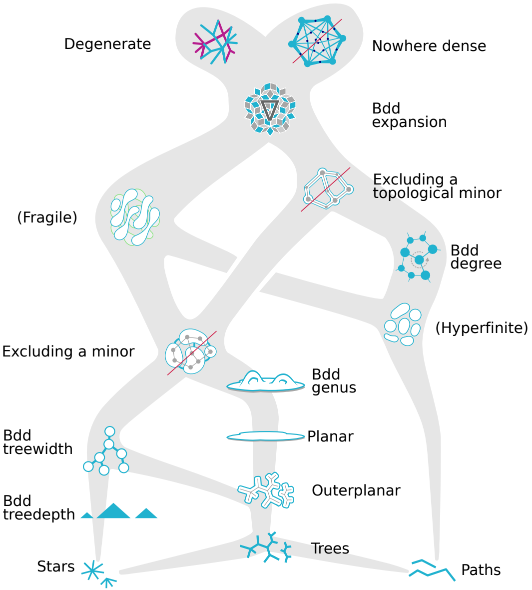


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**Thank you for listening!**