Sparsity

tutorial at PCC'20

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Faculty of Mathematics, Informatics, and Mechanics
University of Warsaw

September 17th, 2020

Organization

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Rough plan:

9:15 – 10:00	**Introduction**
10:15 – 11:00	**Generalized coloring numbers**
11:15 – 12:00	**Treedepth and low treedepth colorings**
12:15 – 13:00	**Uniform quasi-wideness and ladders**

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Format:

– Lecture interleaved with short exercises. $\rightsquigarrow$ **Be active!**

– Understanding checks by writing **+1** in the chat.

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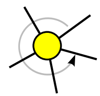

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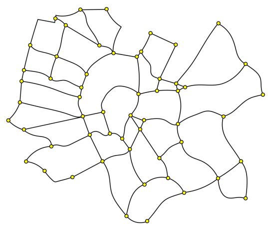

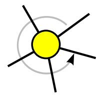

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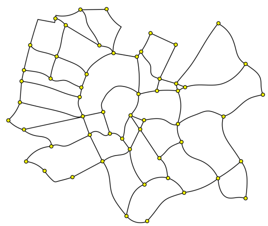

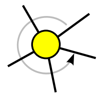

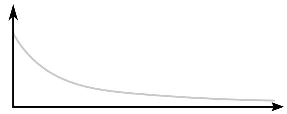

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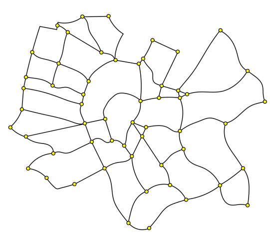

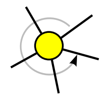

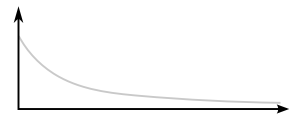

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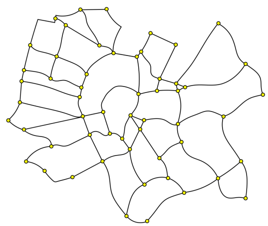

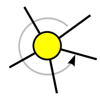

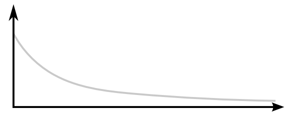

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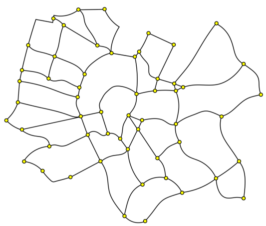

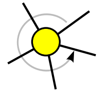

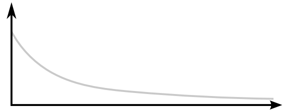

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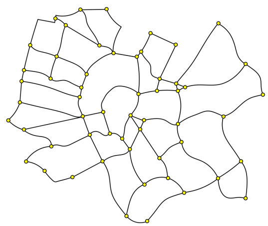

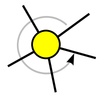

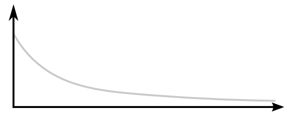

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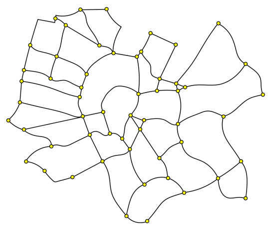

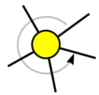

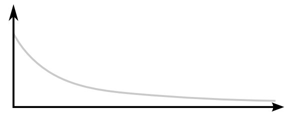

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Sparsity: a young area of graph theory that $\pm$ achieves all of the above.

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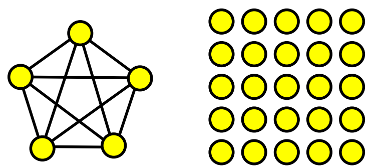

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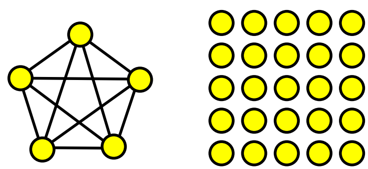

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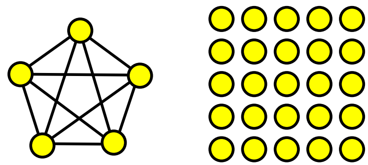

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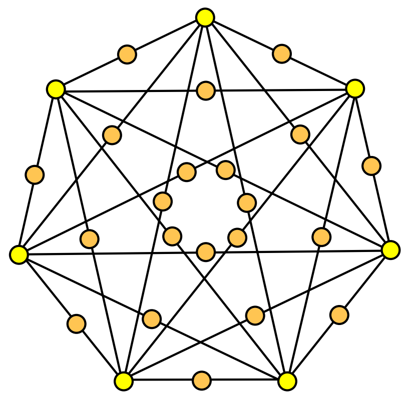

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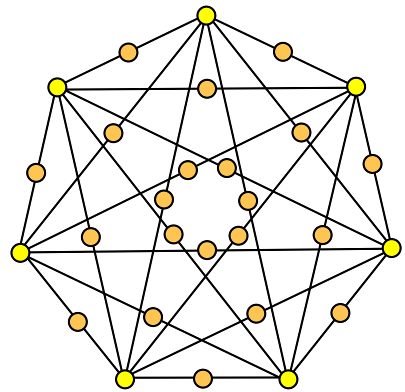

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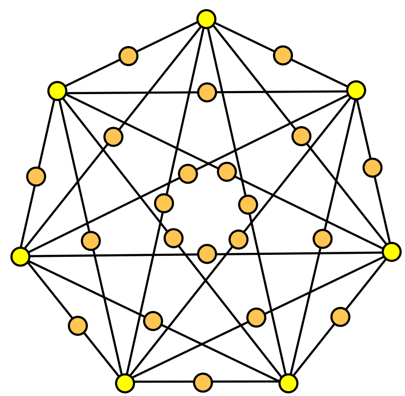
– Is this graph really **sparse**?



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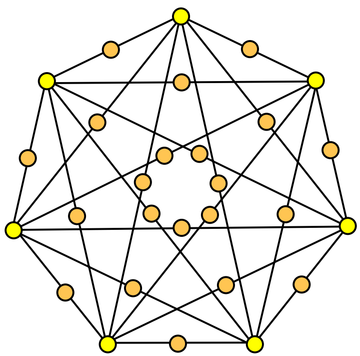

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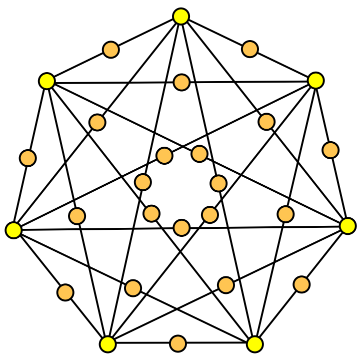

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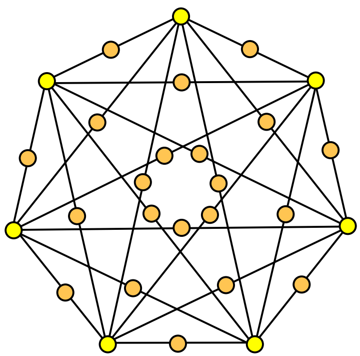

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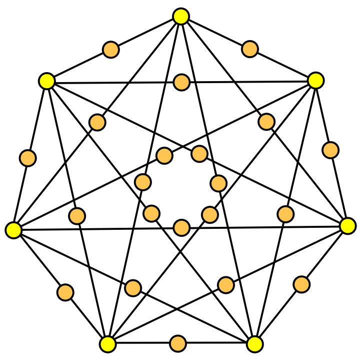

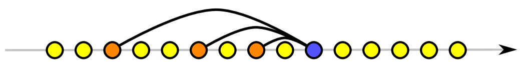

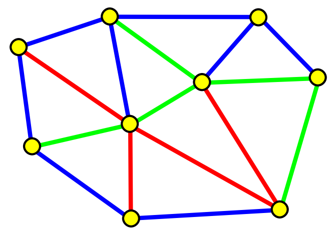

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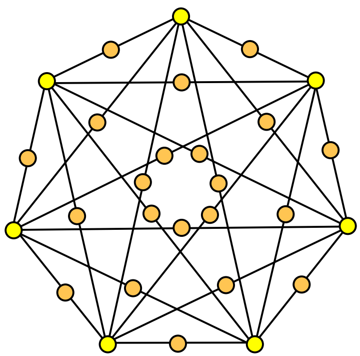

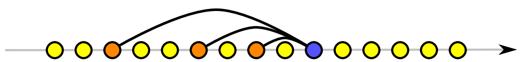

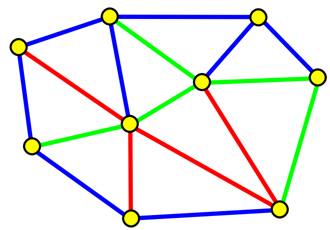

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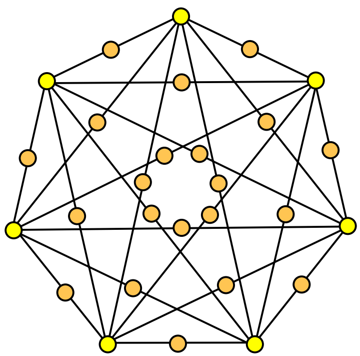

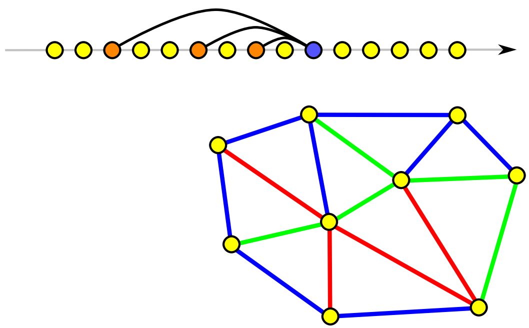

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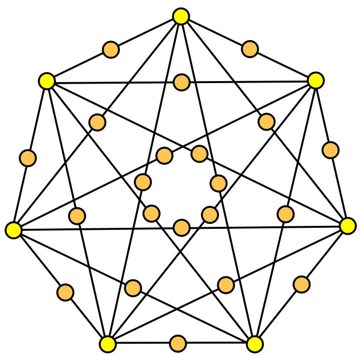
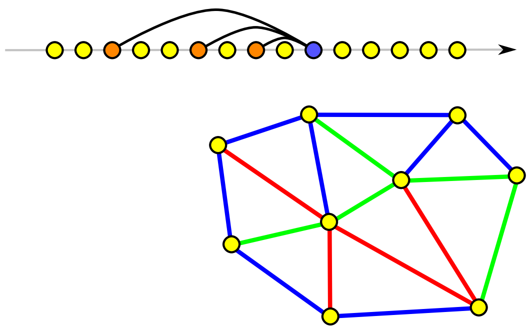
– **Reason:** It contains a dense substructure visible at "depth" 1.





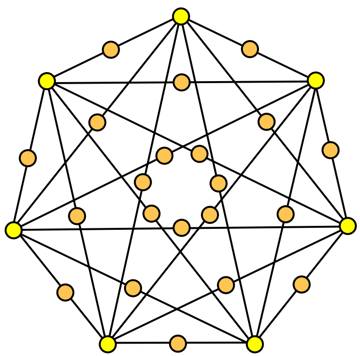
Measuring sparsity

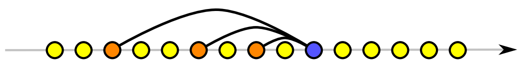
Option 1. We decide that a subdivided complete graph is **sparse**.

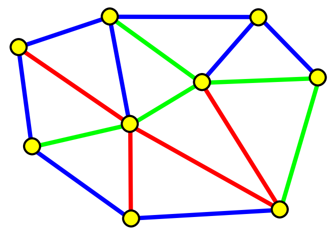
- We can construct a theory around the parameter $\mathrm{mad}(\cdot)$.
- $\mathrm{mad}(\cdot)$ is essentially equivalent to **arboricity** and **degeneracy**.
- These connections are useful, but not really very deep.

Option 2. We decide that a subdivided complete graph is **dense**.

- **Reason:** It contains a dense substructure visible at "depth" 1.
- **Need:** A notion of **embedding** that would capture this.







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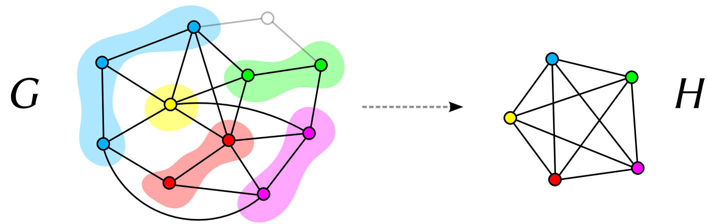

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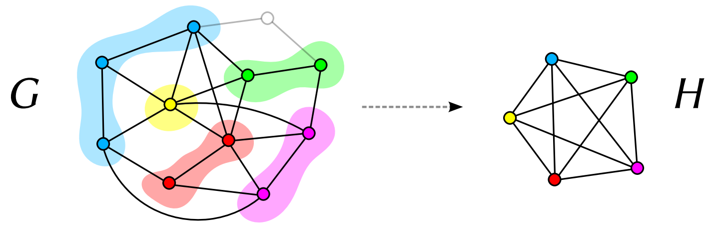

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Planar graphs are exactly $\{K_5, K_{3,3}\}$-minor-free graphs.

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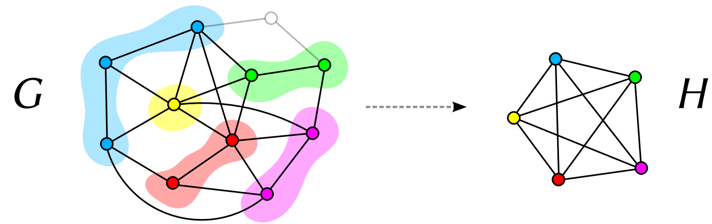

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Theorem (Robertson and Seymour)

For every $t \in \mathbb{N}$, every K_t-minor-free graph looks like this:

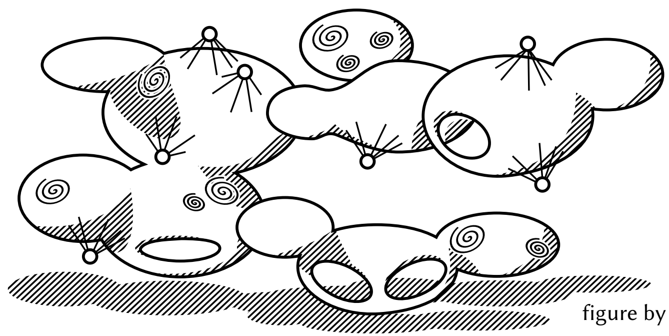

figure by Felix Reidl

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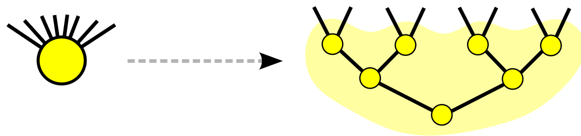

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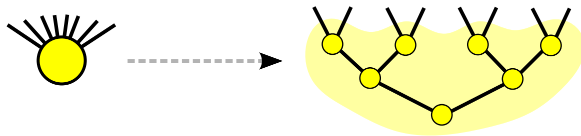

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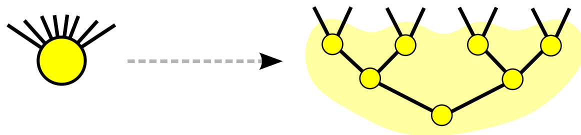

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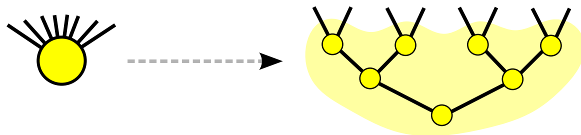

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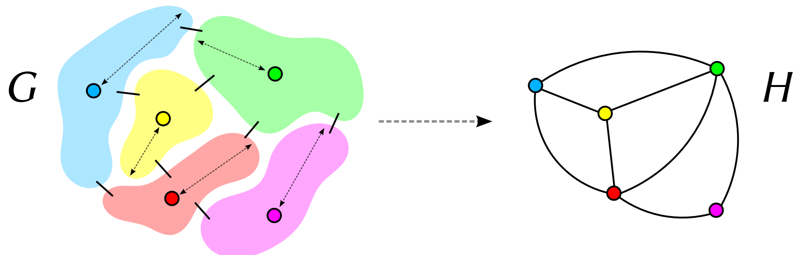

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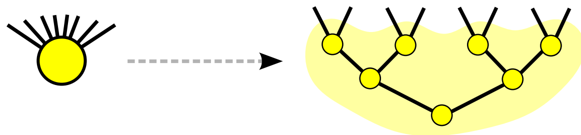

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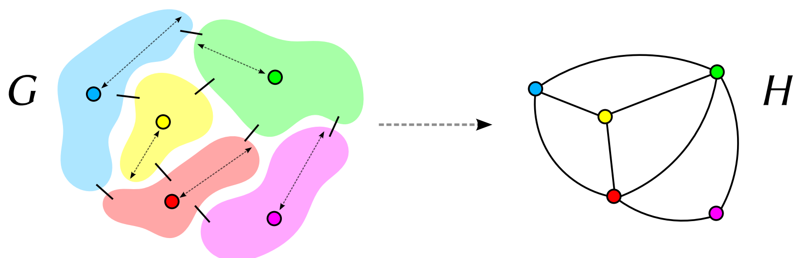

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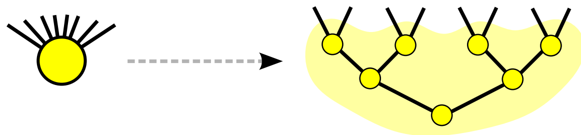

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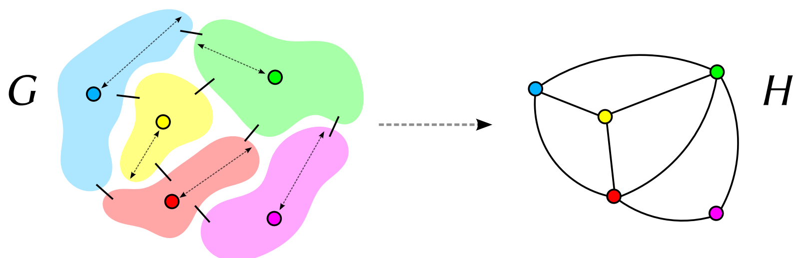

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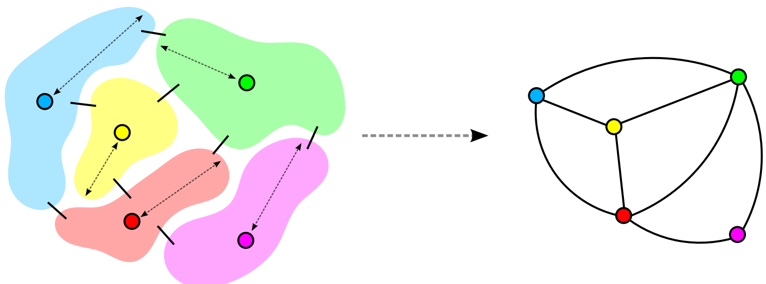

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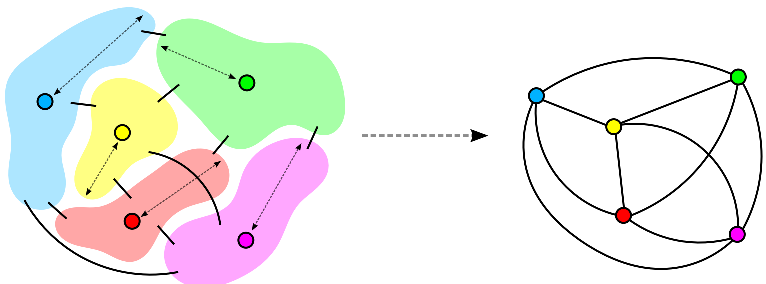

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- **Classes** of graphs as basic objects of interest.

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Fact. $\mathcal{C}$ is **nowhere dense** $\Rightarrow \forall d, \varepsilon > 0$, $\nabla_d(G) \leqslant \mathcal{O}(n^{\varepsilon})$ for all $G \in \mathcal{C}$.

The World of Sparsity

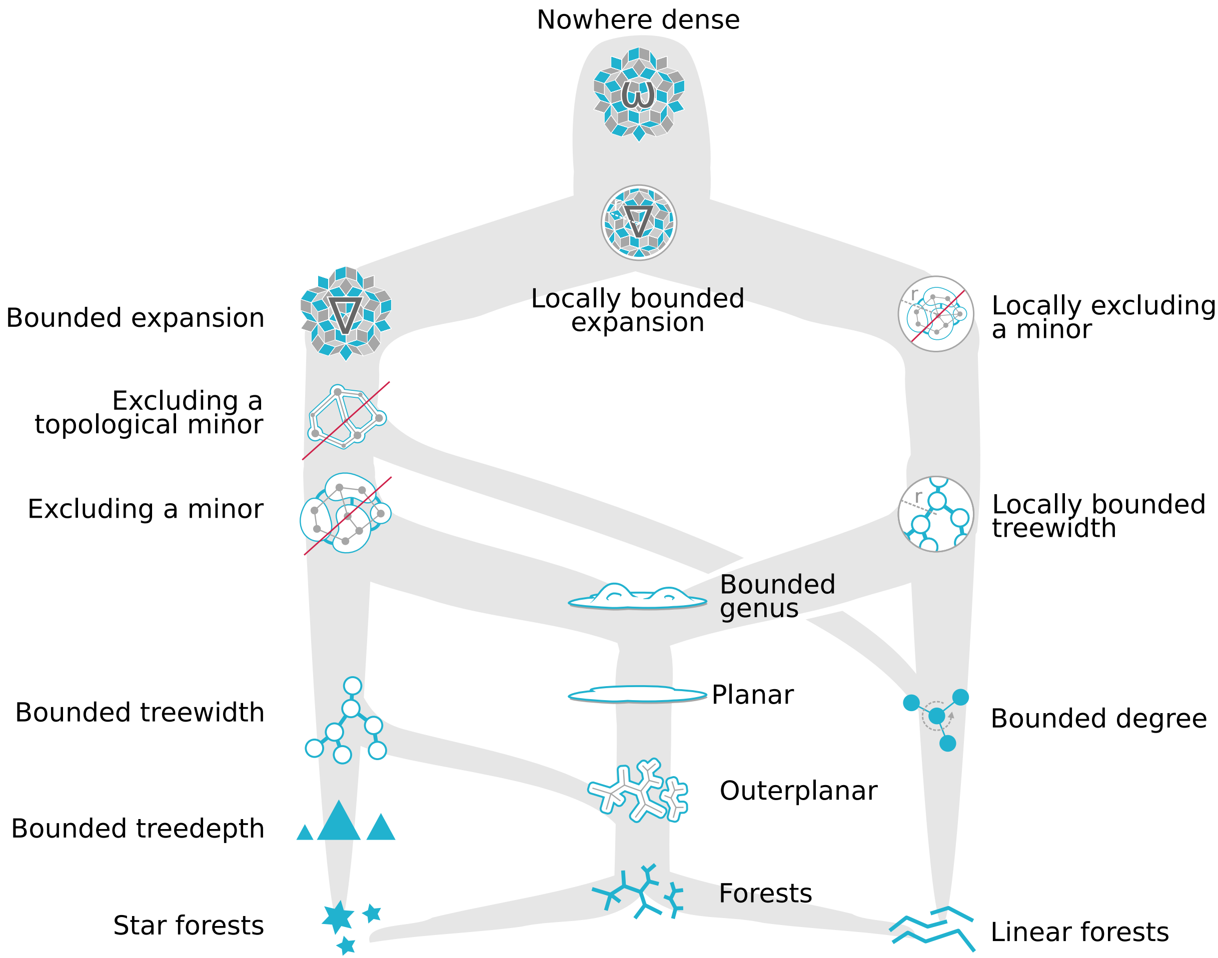
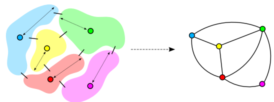


figure by Felix Reidl

Equivalent characterizations

Sparsity of shallow minors



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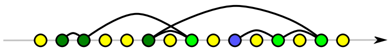

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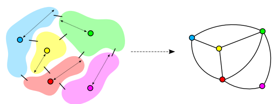
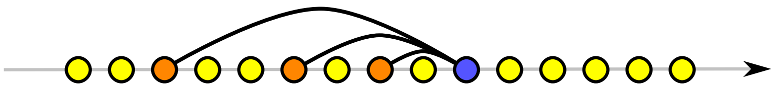
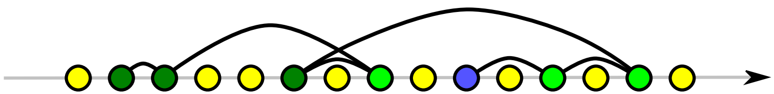




Degeneracy



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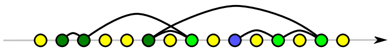

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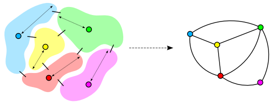
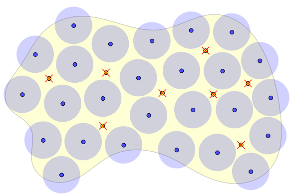
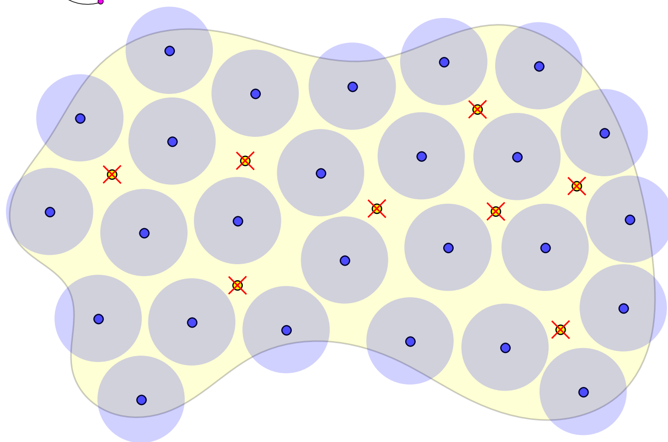


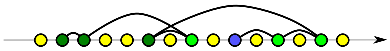
Uniform quasi-wideness

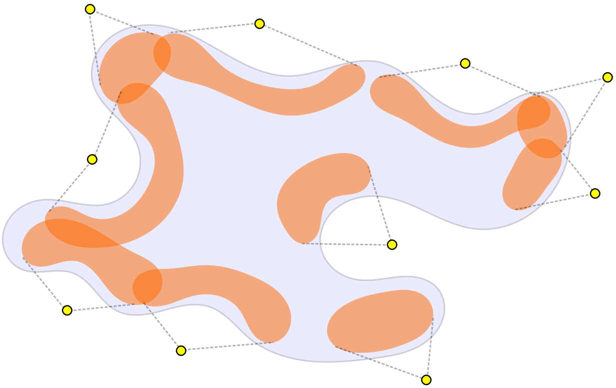




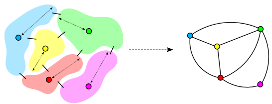
Equivalent characterizations

Generalized coloring numbers





Sparsity of shallow minors



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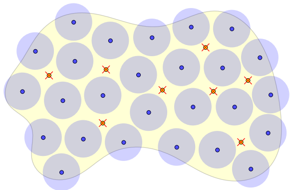
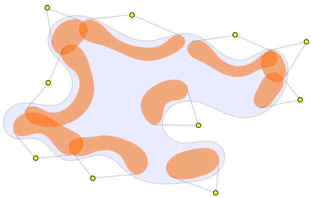


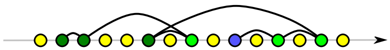
Neighborhood complexity



Equivalent characterizations

Part 2

Generalized coloring numbers



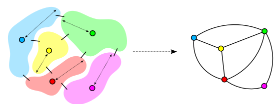
Stability

Sparsity of shallow top-minors

Fraternal augmentations

Neighborhood covers

Sparsity of shallow minors



Part 3

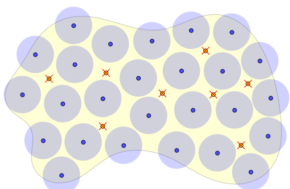
Low treedepth colorings

Uniform quasi-wideness

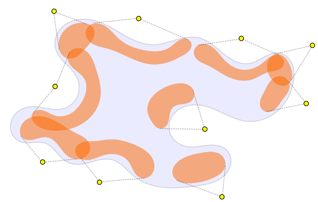
k-Helly property

Neighborhood complexity

Part 4



Splitter game



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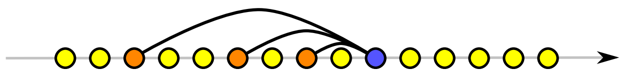

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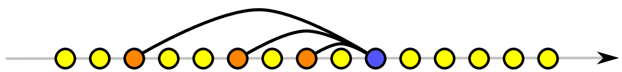

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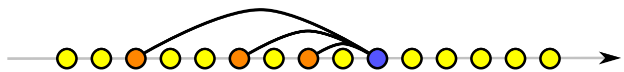

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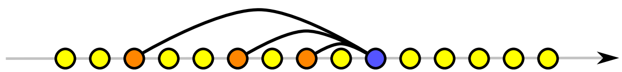

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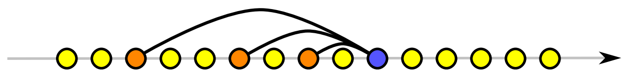

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Sol: Greedy left-to-right coloring on the ordering.

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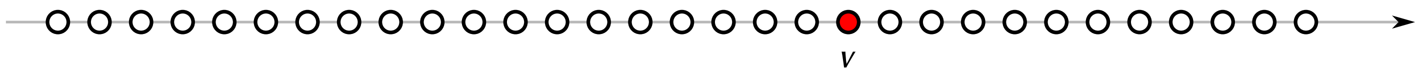

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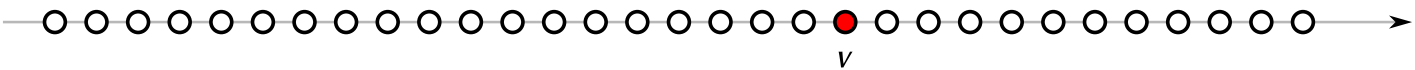

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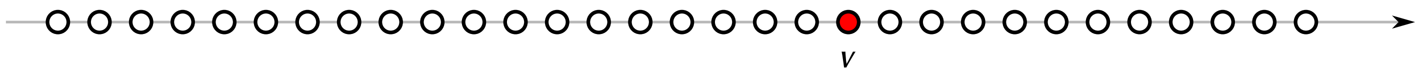
Want: Define "σ-smaller neighbors" of v at "depth" d.



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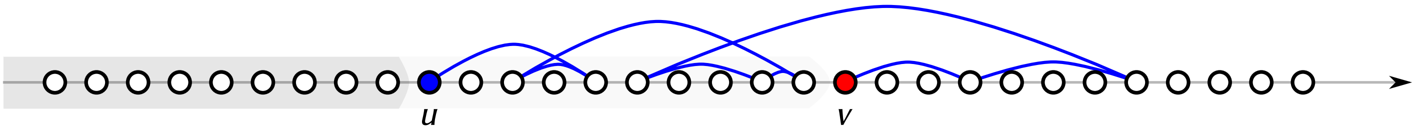
D1: $u \leqslant_\sigma v$ is **weakly d-reachable** from v $\Leftrightarrow$

There is a v-to-u path P of length $\leqslant d$ that is entirely $\geqslant_\sigma u$.



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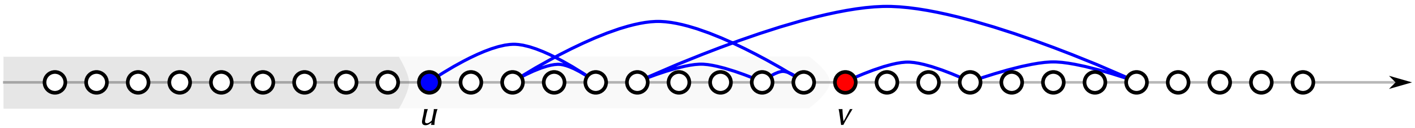

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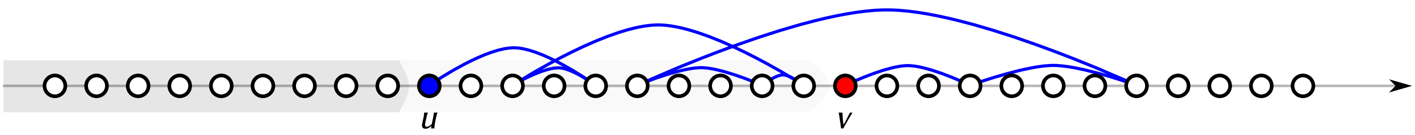

D2: $u \leqslant_\sigma v$ is **strongly d-reachable** from v $\Leftrightarrow$

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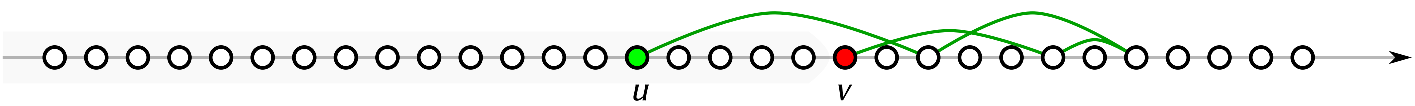

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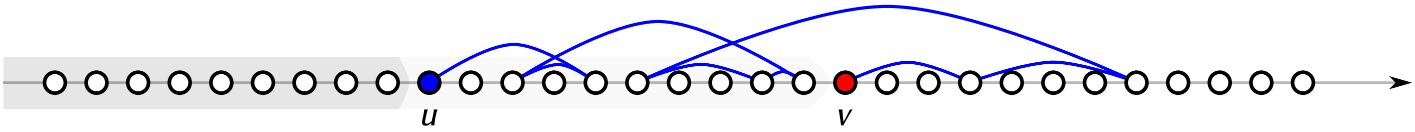

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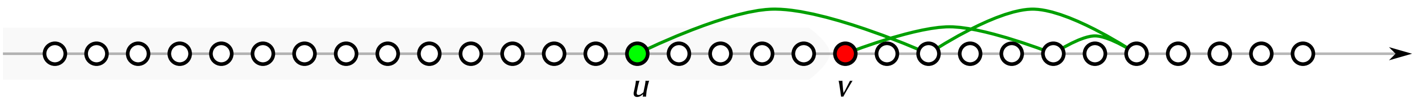

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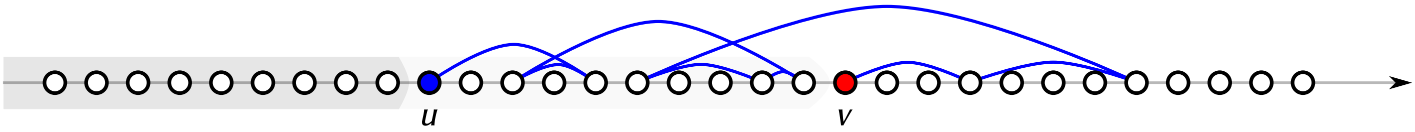

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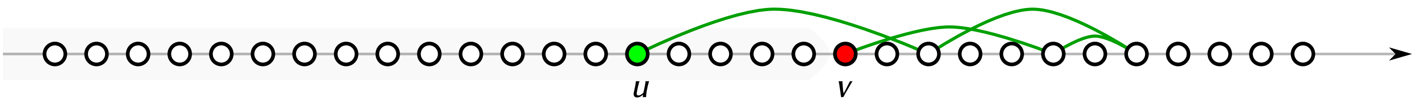
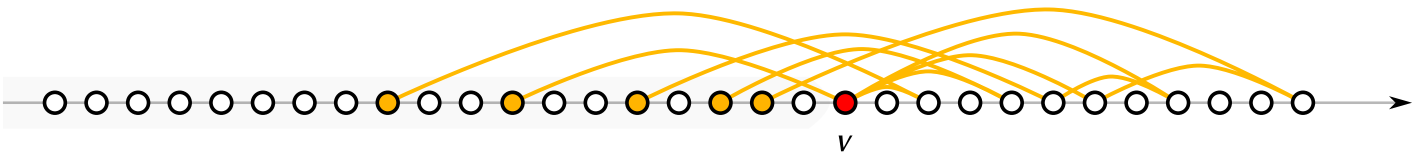


D3: $\mathrm{adm}_d(G, \sigma, v)$:=max # of disjoint v-to-($<_\sigma v$) paths of length $\leqslant d$



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For a graph G and vertex ordering σ, we define:

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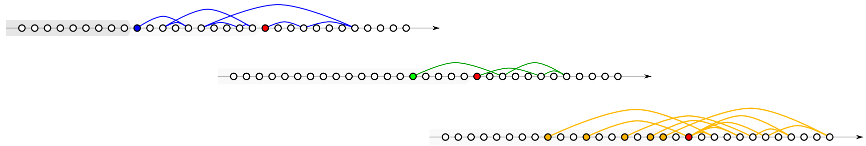
1. $\mathrm{dgn}(G) = \mathrm{adm}_0(G) = \mathrm{scol}_0(G) - 1 = \mathrm{wcol}_0(G) - 1.$

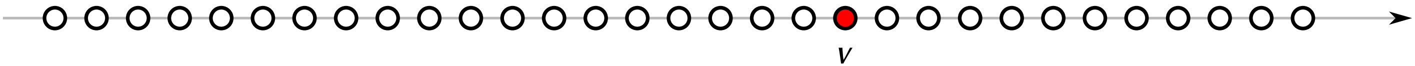
2. $\mathrm{adm}_d(G) \leqslant \mathrm{scol}_d(G) \leqslant \mathrm{wcol}_d(G).$

Now: These parameters are functionally equivalent.

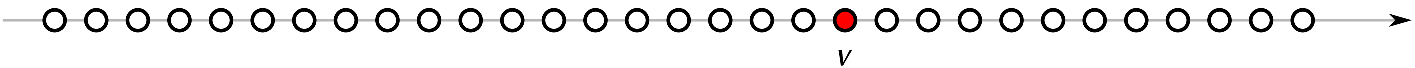
Equivalence of generalized coloring numbers



1. $\mathrm{scol}_d(G) \leqslant 1 + \mathrm{adm}_d(G)^d$.



2. $\mathrm{wcol}_d(G) \leqslant 1 + \mathrm{scol}_d(G) + \mathrm{scol}_d(G)^2 + \ldots + \mathrm{scol}_d(G)^d$.



Equivalence with grads

Lemma

For a graph G and $d \in \mathbb{N}$, we have:

$$\mathrm{adm}_d(G) \leqslant 6d(\nabla_d(G) + 1)^3, \qquad \nabla_d(G) \leqslant \mathrm{wcol}_{4d+1}(G).$$

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Proof of $\nabla_d(G) \leqslant \mathrm{wcol}_{4d+1}(G)$:

- Let $H \preceq_d G$ and $\{J_u \colon u \in V(H)\}$ be a model.
- Let $\phi(u) \coloneqq \min_\sigma V(J_u)$.
- Let $w \in V(H)$ be such that $\phi(w)$ is σ-maximal.
- **Obs:** For each $u \in N_H(w)$, we have $\phi(u) \in \mathrm{WReach}_{4d+1}[G, \sigma, \phi(w)]$.
- **Cor:** w has degree $\leqslant \mathrm{wcol}_{4d+1}(G)$ in H.
- **Cor:** Every $H \preceq_d G$ has a vertex of degree $\leqslant \mathrm{wcol}_{4d+1}(G)$. $\square$

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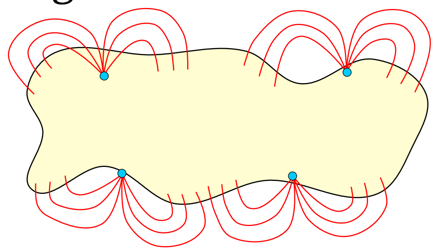

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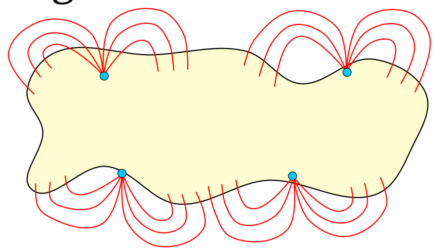

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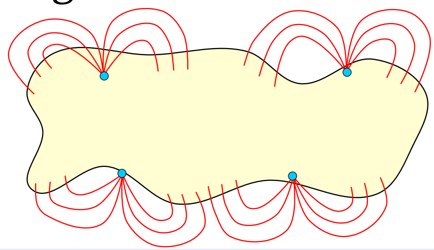

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Theorem

For a class of graphs $\mathcal{C}$, the following are equivalent:

– $\mathcal{C}$ has bounded expansion;

– $\nabla_d(\mathcal{C})$ is finite for all $d \in \mathbb{N}$;

– $\mathrm{wcol}_d(\mathcal{C})$ is finite for all $d \in \mathbb{N}$;

– $\mathrm{scol}_d(\mathcal{C})$ is finite for all $d \in \mathbb{N}$;

– $\mathrm{adm}_d(\mathcal{C})$ is finite for all $d \in \mathbb{N}$.

Distance-d domination

Definition

Let G be a graph and $d \in \mathbb{N}$.

$D \subseteq V(G)$ is a **dist-d dominating set** if $\bigcup_{u \in D} \mathrm{Ball}_d(u) = V(G)$.

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1. Prove that $|D| \leqslant \mathrm{wcol}_{2d}(G, \sigma) \cdot \mathrm{dom}_d(G)$.

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Cor: Constant-factor apx algorithm for $\mathrm{dom}_d(G)$ in bnd expansion classes.

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Proof: A greedy procedure on a vertex ordering witnessing $\mathrm{wcol}_{2d}(G)$.

Part 3:

Treedepth and low treedepth colorings

Treedepth

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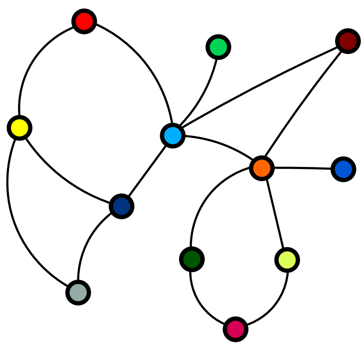
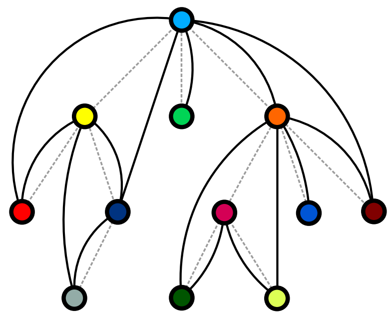
Elimination forest of G is a rooted forest F with $V(F) = V(G)$ s.t:

$$u, v \text{ adjacent in } G \quad \Rightarrow \quad u, v \text{ in ancestor/descendant relation in } F$$





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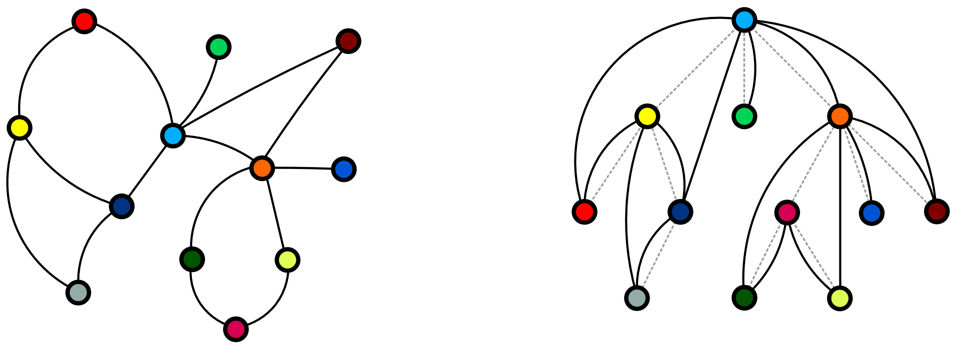

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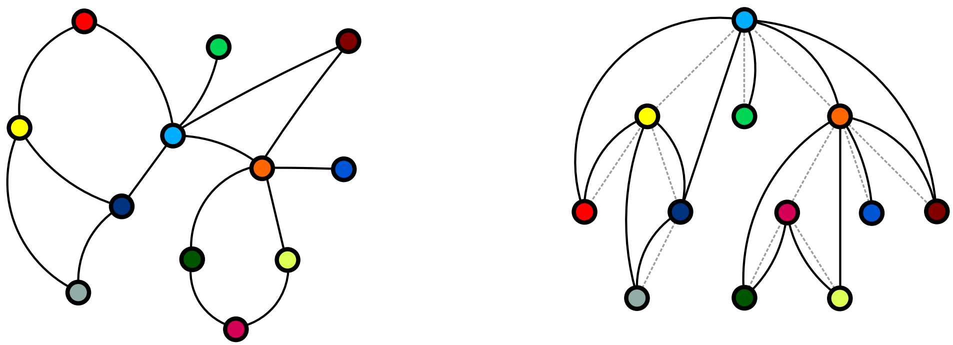

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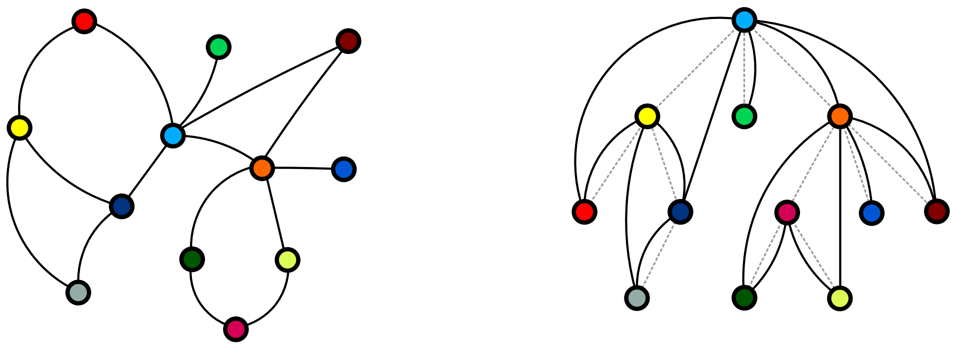

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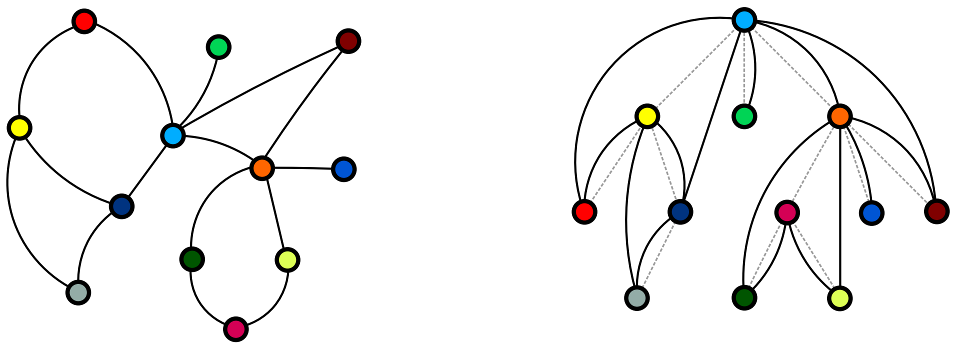

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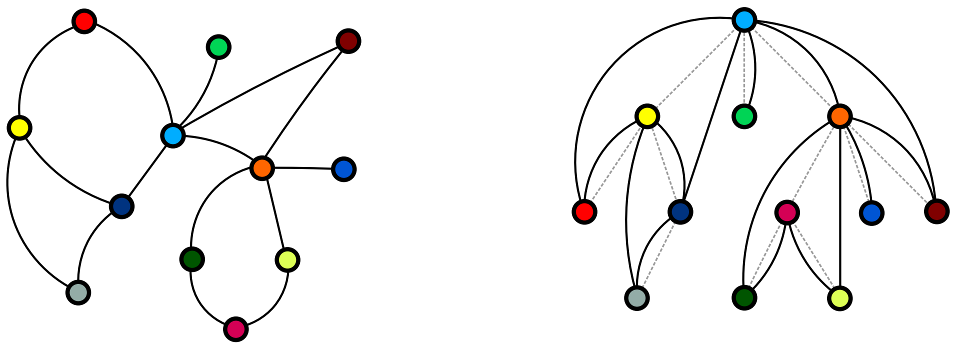

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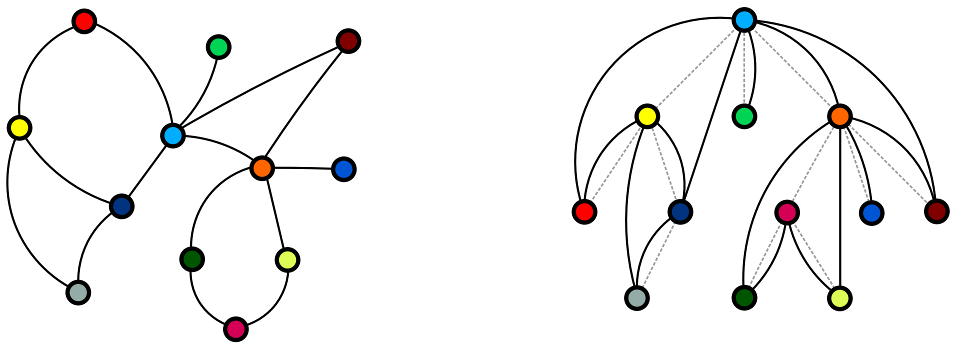

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1. Compute:

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Let ϕ be the greedy coloring with $\mathrm{wcol}_{2p-1}(G)$ colors s.t.:

For each v, $\phi(v) \notin$ colors given to $\mathrm{WReach}_{2p-1}(G) \setminus \{v\}$ by ϕ.

Constructing a low td coloring

1. P is a path on 2^{p-1} vertices $\quad\Rightarrow\quad$ P receives $\geqslant p$ different colors.

2. $H \subseteq G$ is connected and receives $\leqslant p$ colors $\quad\Rightarrow$

H has a vertex of unique color.

3. $H \subseteq G$ receives $\leqslant p$ colors $\quad\Rightarrow\quad$ $\mathrm{td}(H) \leqslant p$.

Low td colorings

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Theorem (low td colorings)

Let $\mathcal{C}$ be a class of **bnd expansion** and $p \in \mathbb{N}$. Then there is $M(p) \in \mathbb{N}$ such that every $G \in \mathcal{C}$ has a coloring with $M(p)$ colors satisfying:

Every p colors together induce a subgraph of treedepth $\leqslant p$.

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- for each $X \subseteq V(G)$ with $|X| \leqslant p$, there is A_i such that $X \subseteq A_i$.

Proof:

- $\{A_1, A_2, \ldots, A_{N(p)}\} =$ subsets of p colors.

Low td colorings

Theorem (low td colorings)

Let $\mathcal{C}$ be a class of **bnd expansion** and $p \in \mathbb{N}$. Then there is $M(p) \in \mathbb{N}$ such that every $G \in \mathcal{C}$ has a coloring with $M(p)$ colors satisfying:

Every p colors together induce a subgraph of treedepth $\leqslant p$.

Note: In our proof, $M(p) = \mathrm{wcol}_{2p-1}(\mathcal{C})$.

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Cor: A linear-time algorithm testing whether $Q \subseteq G$.

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Let $\mathcal{C}$ be a class of **bnd expansion** and $d \in \mathbb{N}$ be **odd**. Then there is $L(d) \in \mathbb{N}$ such that every $G \in \mathcal{C}$ has a coloring ϕ with $L(d)$ colors satisfying:

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Main tool for **nowhere dense** classes: **uniform quasi-wideness**.

Part 4:

Uniform quasi-wideness and ladders

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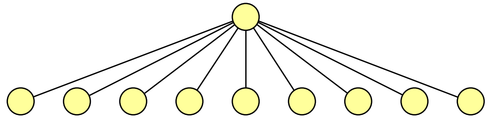

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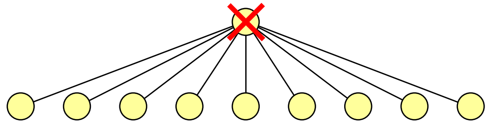

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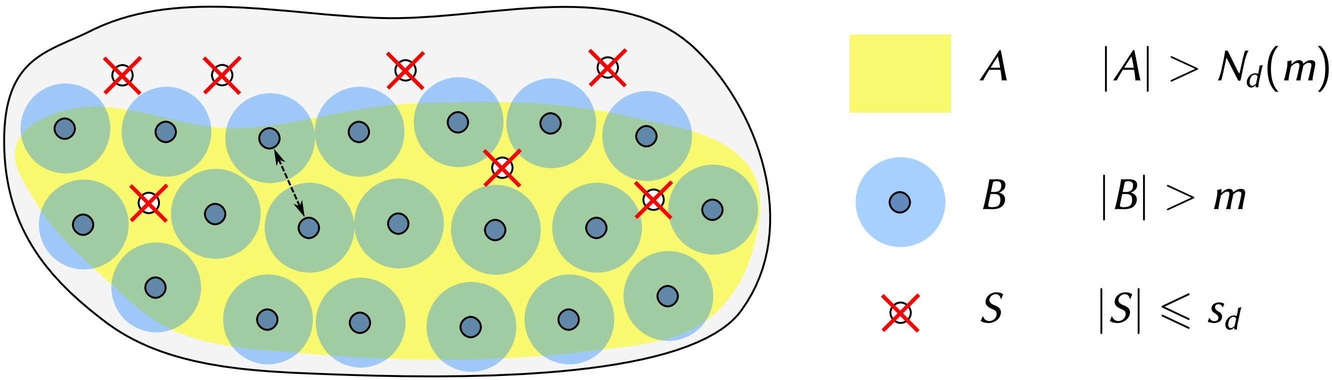

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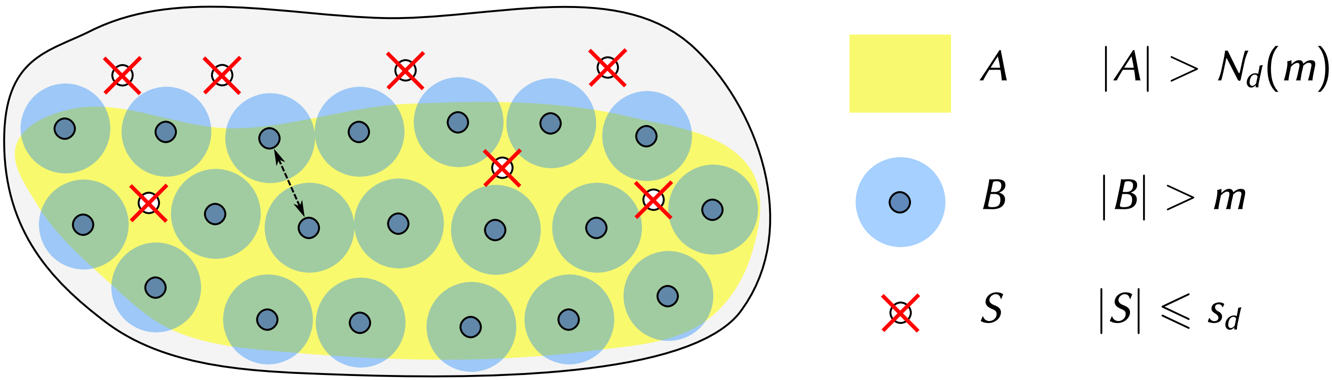

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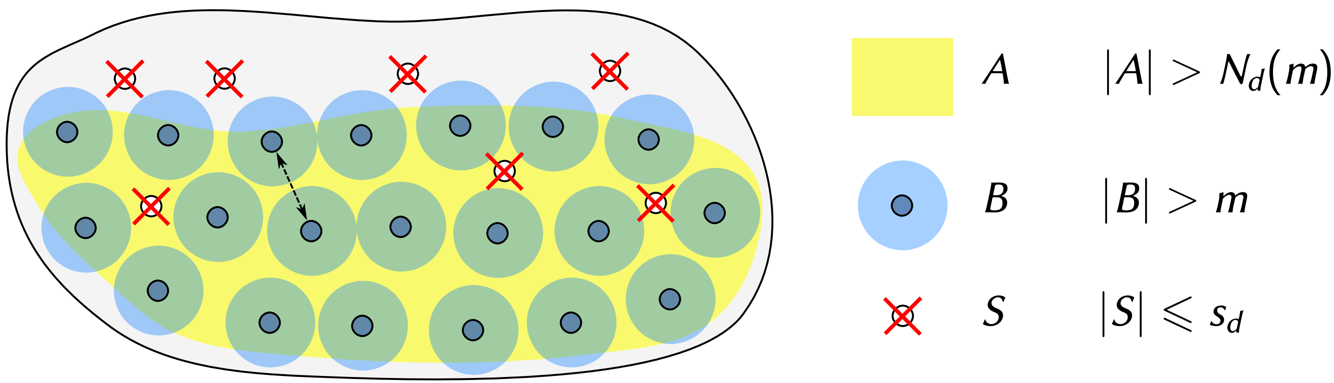

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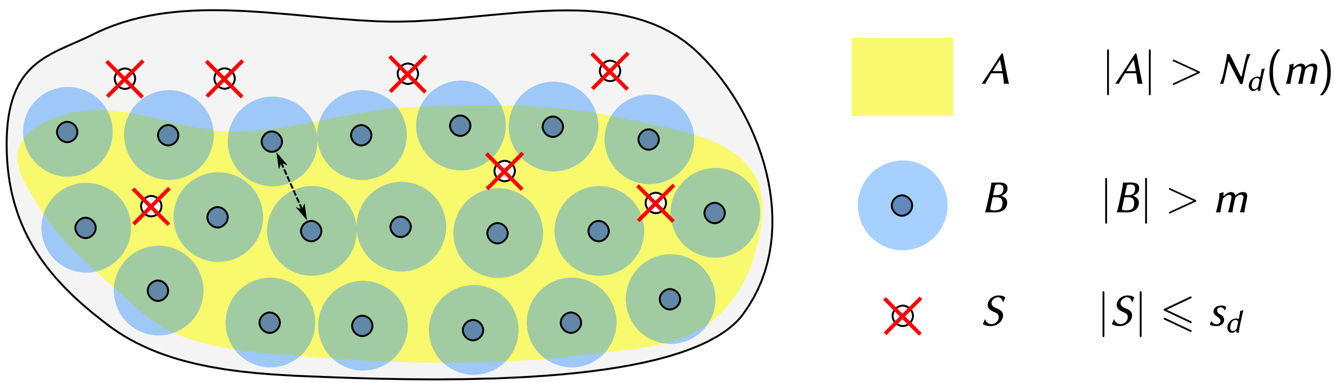

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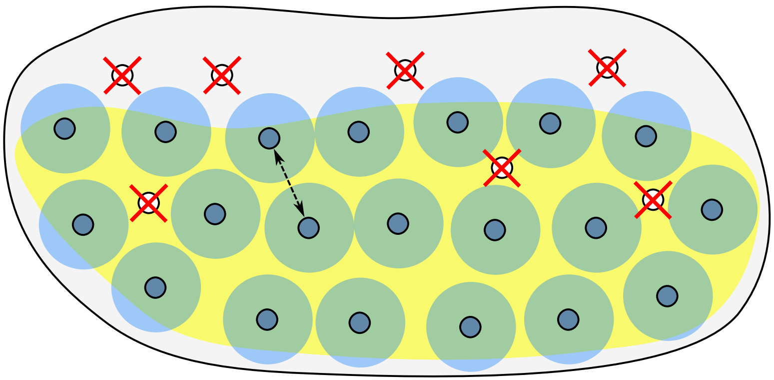

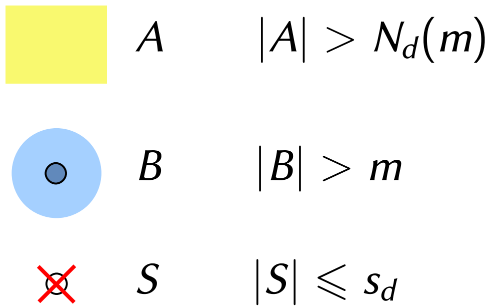

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Example from the work with Fabiański, Siebertz, and Toruńczyk.

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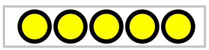
D_1 

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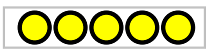
D_1 

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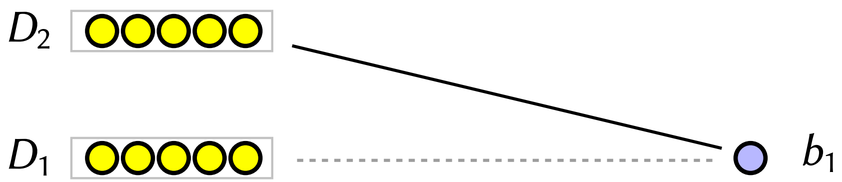

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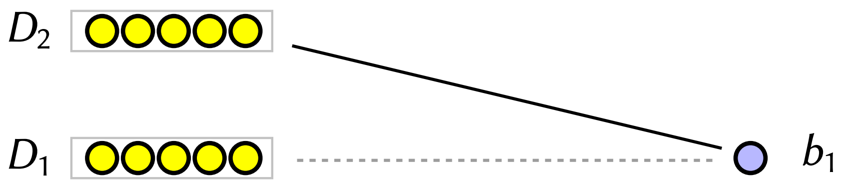

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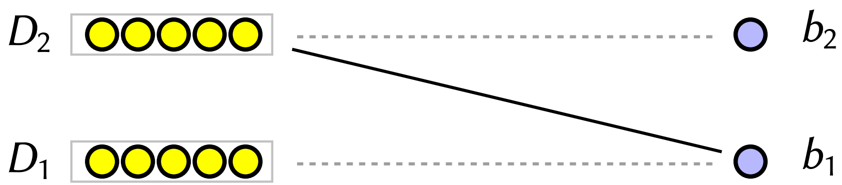

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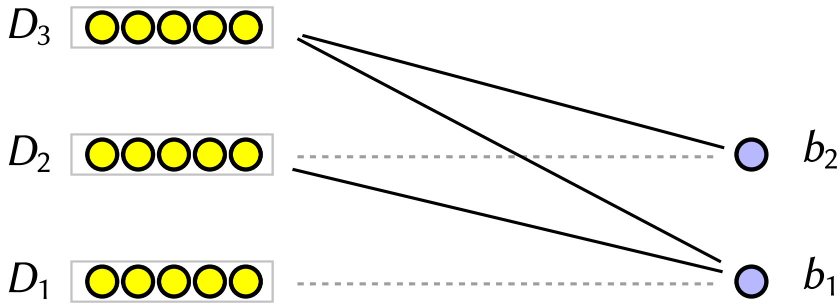

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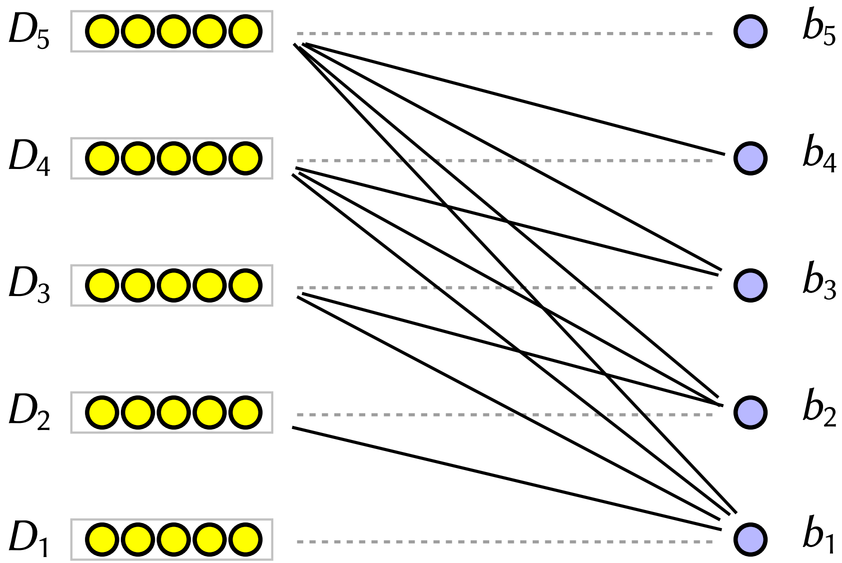

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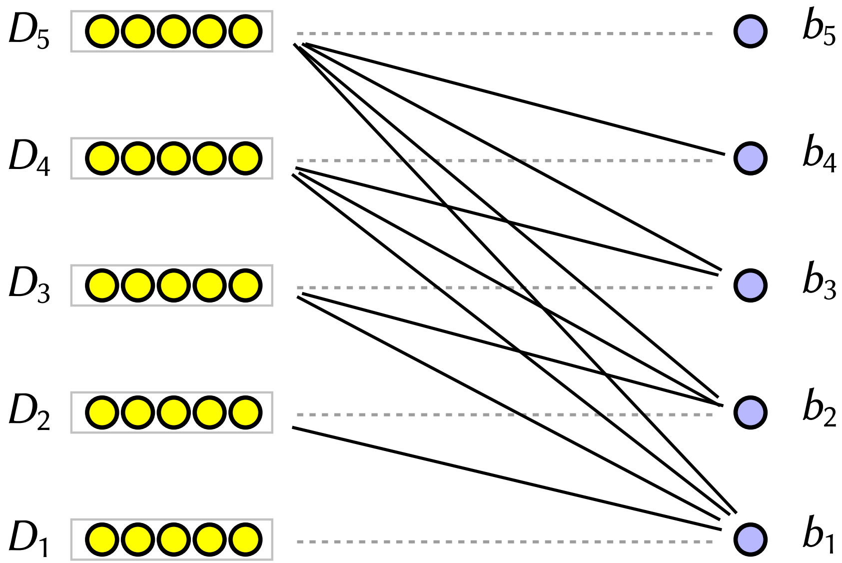

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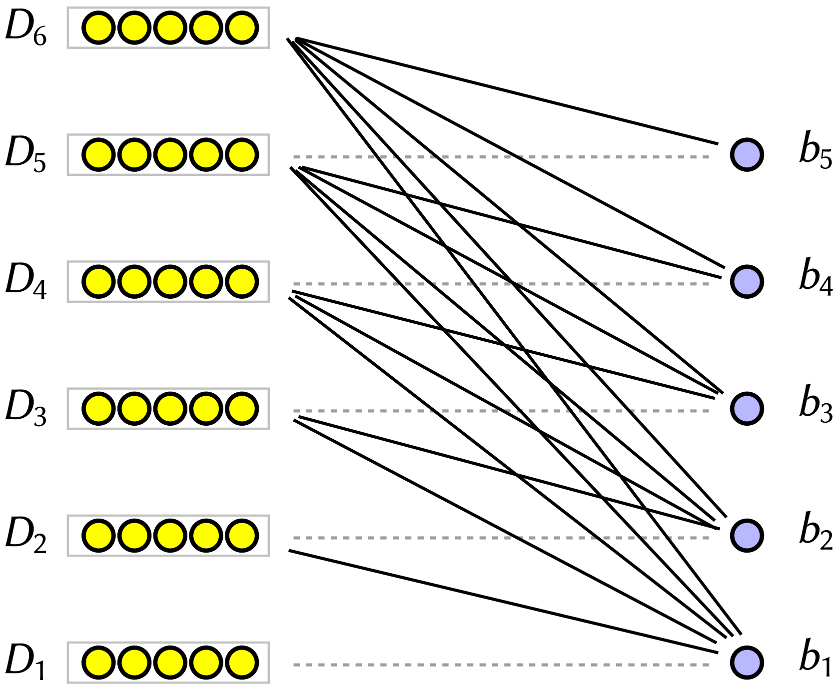

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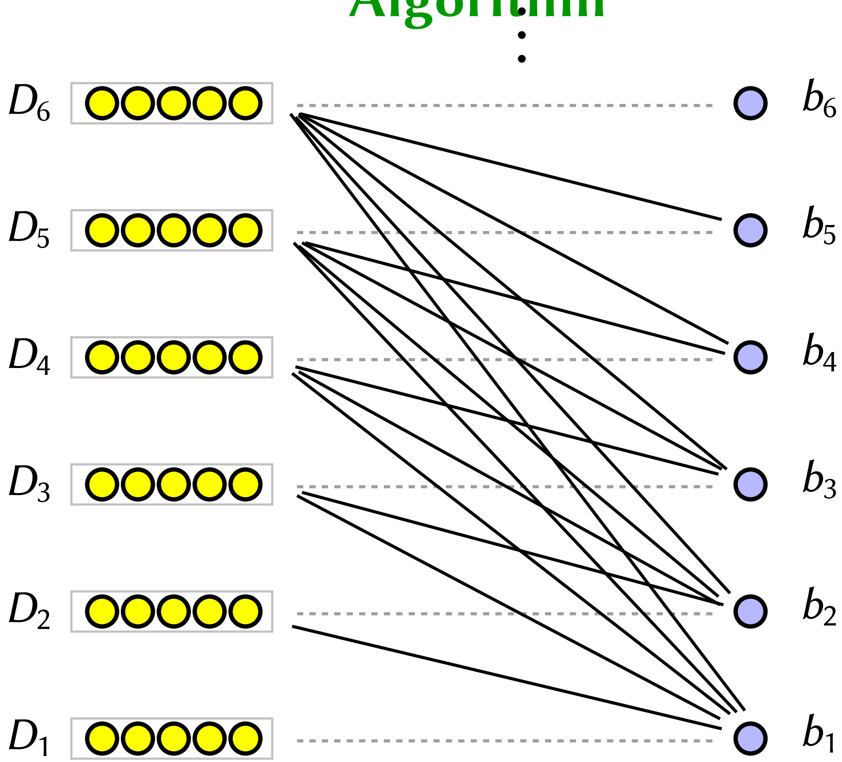

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Now: Proof of the **Lemma** for $k = 1$.

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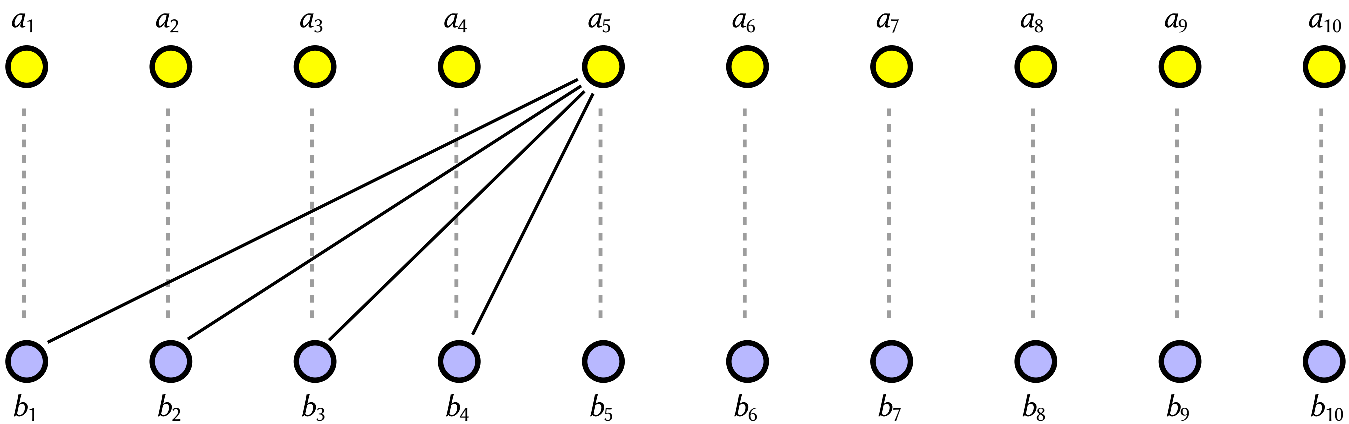

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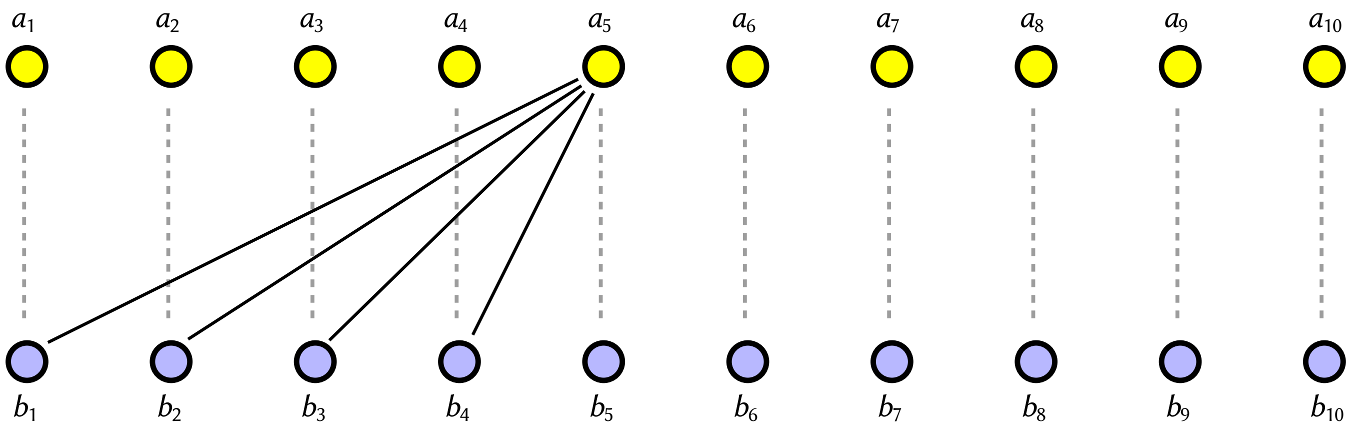

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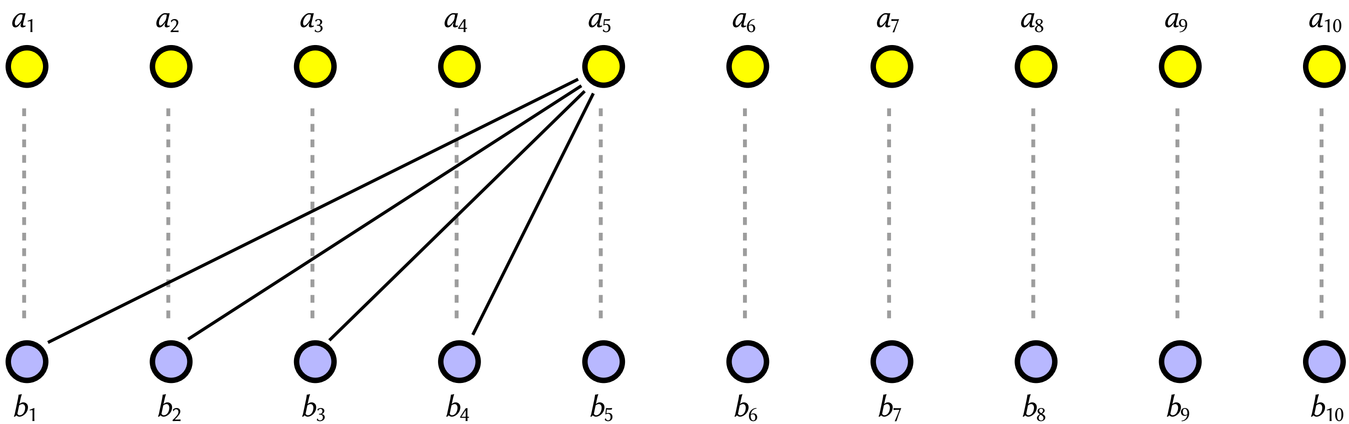

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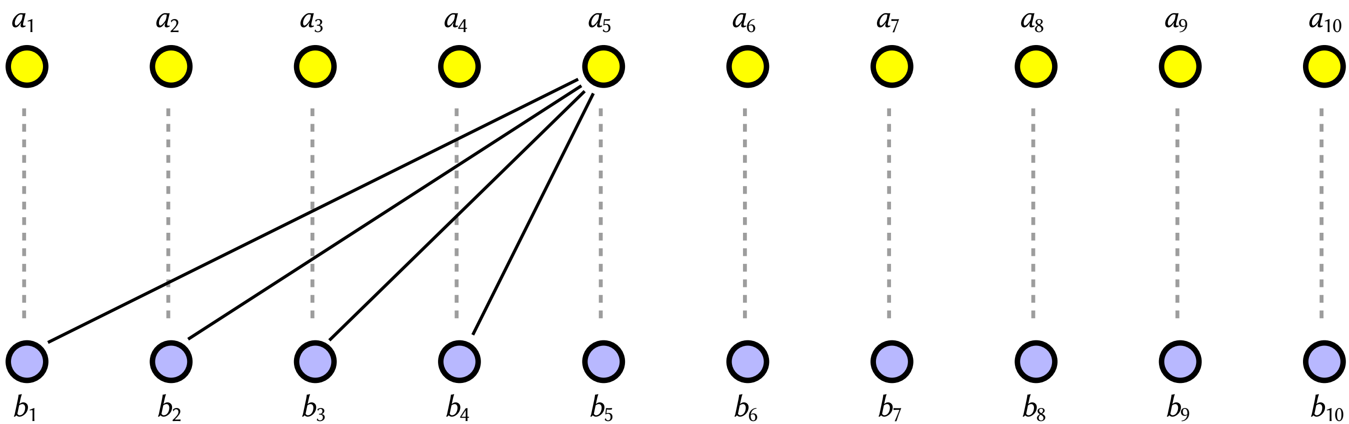

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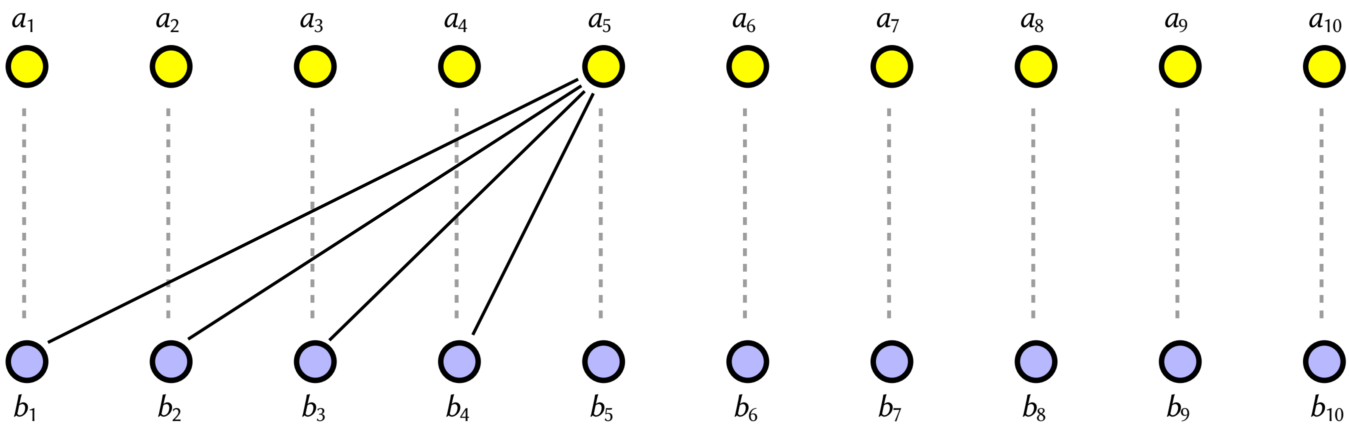
Suppose $\ell > N(2(d+1)^s)$, where $N(\cdot) = N_{2d}(\cdot)$ and $s \coloneqq s_{2d}$.



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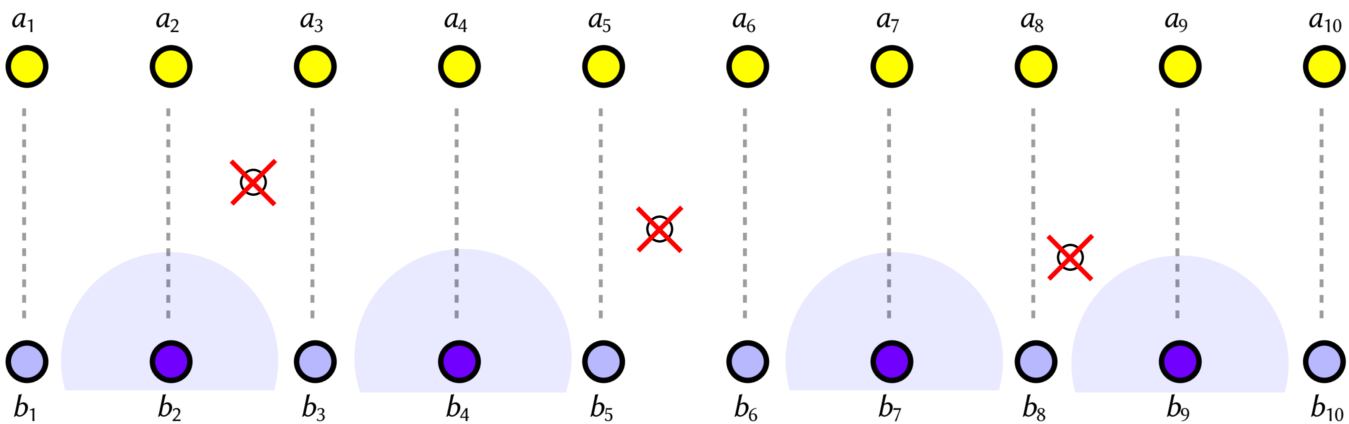

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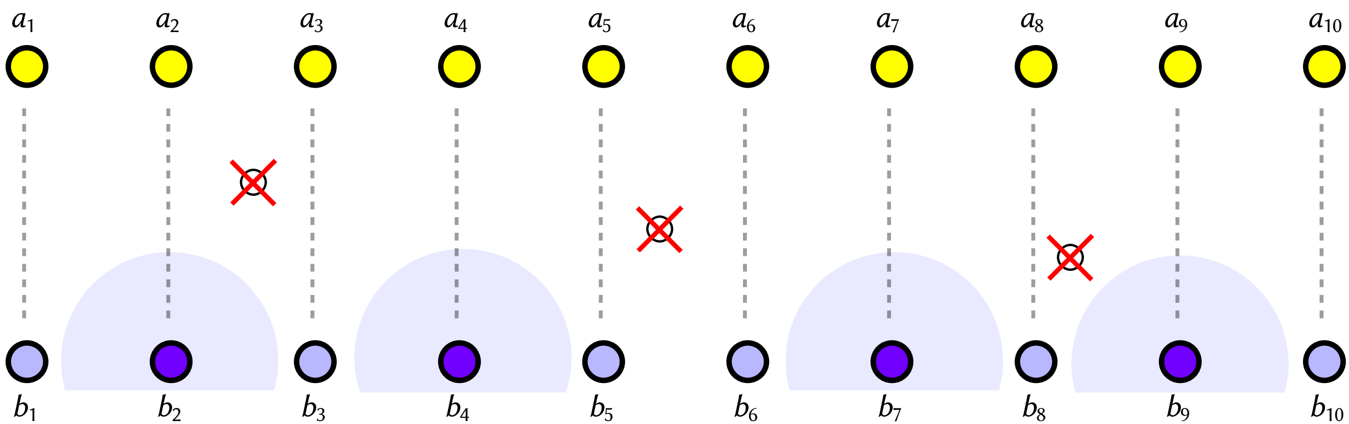

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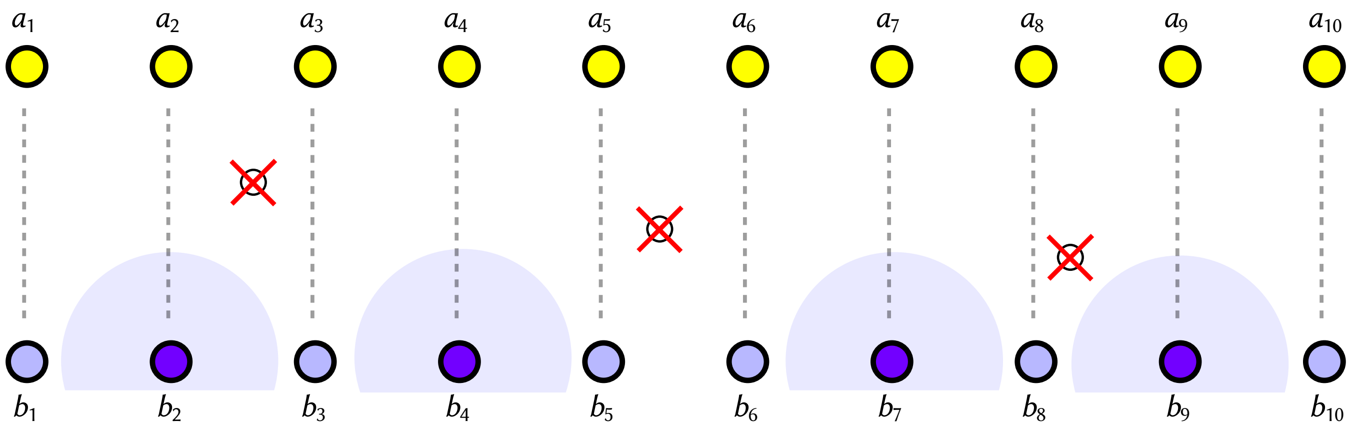

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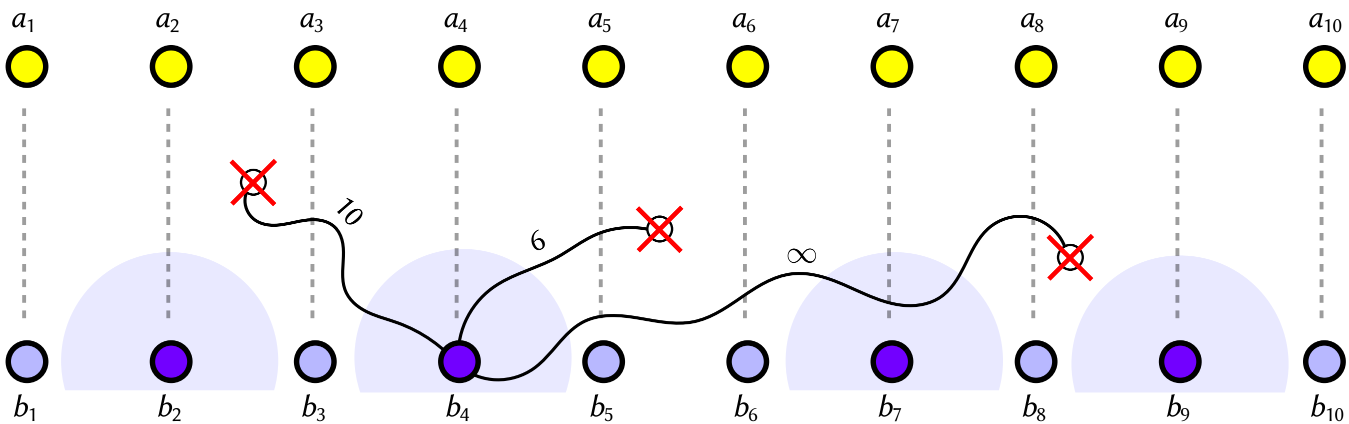

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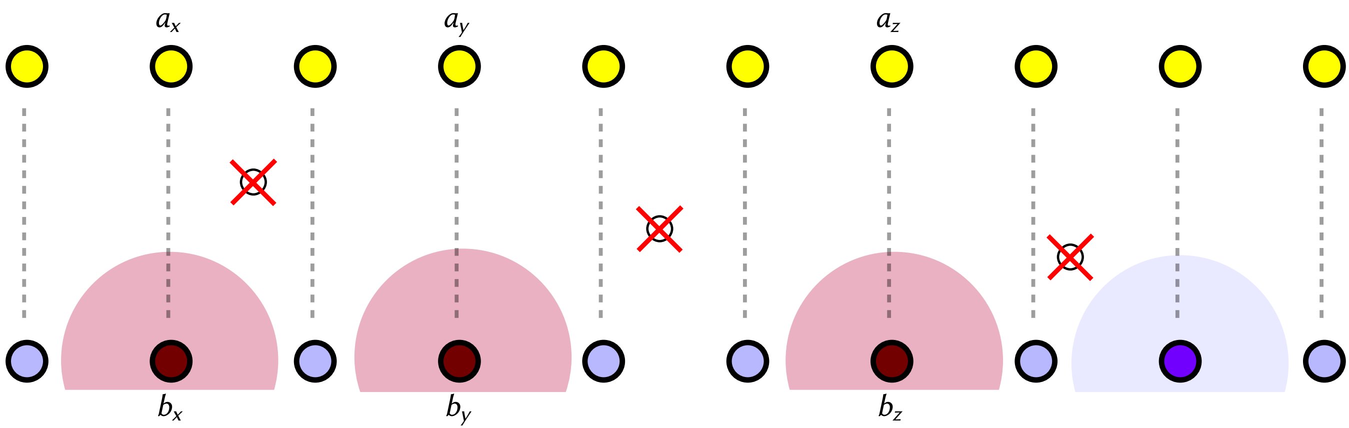
Only $(d+1)^s$ possible profiles $\quad\Rightarrow\quad \exists\ b_x, b_y, b_z$ with same profile.



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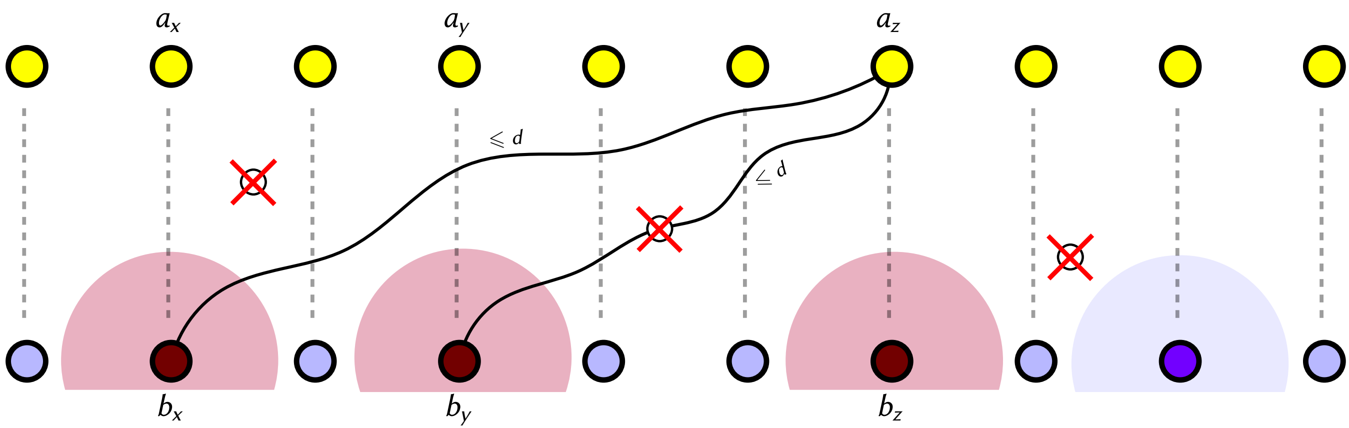

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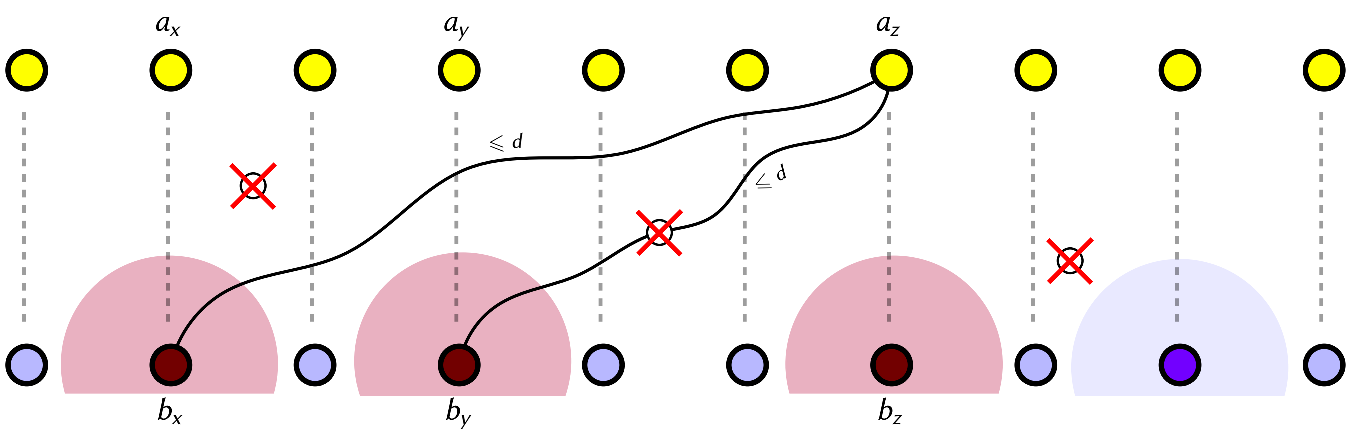

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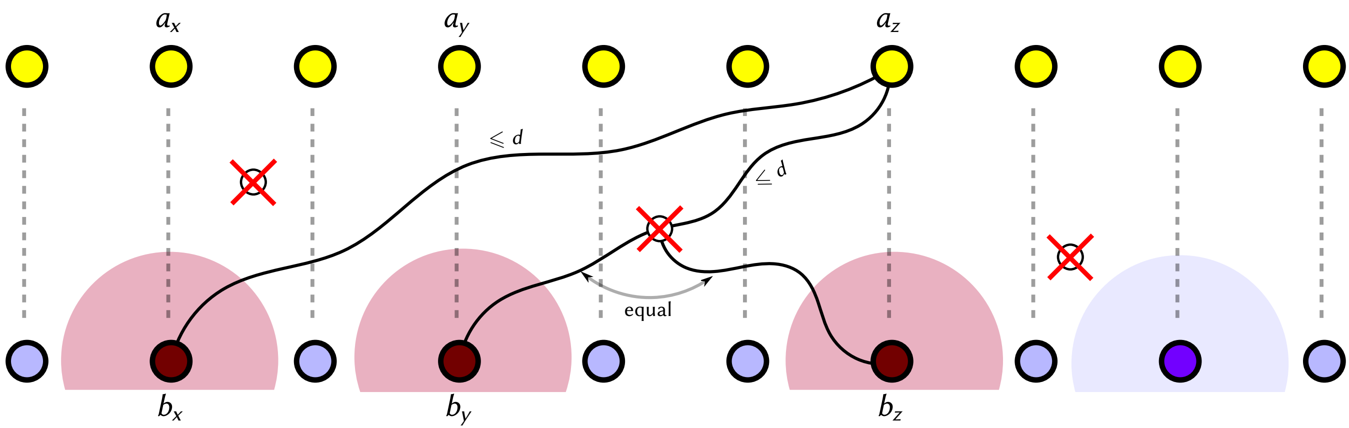

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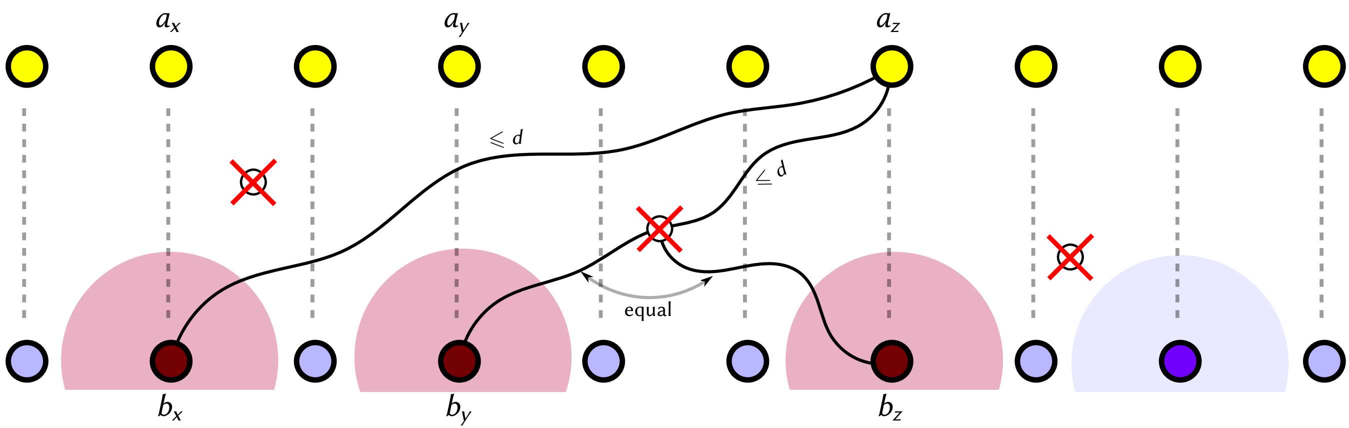

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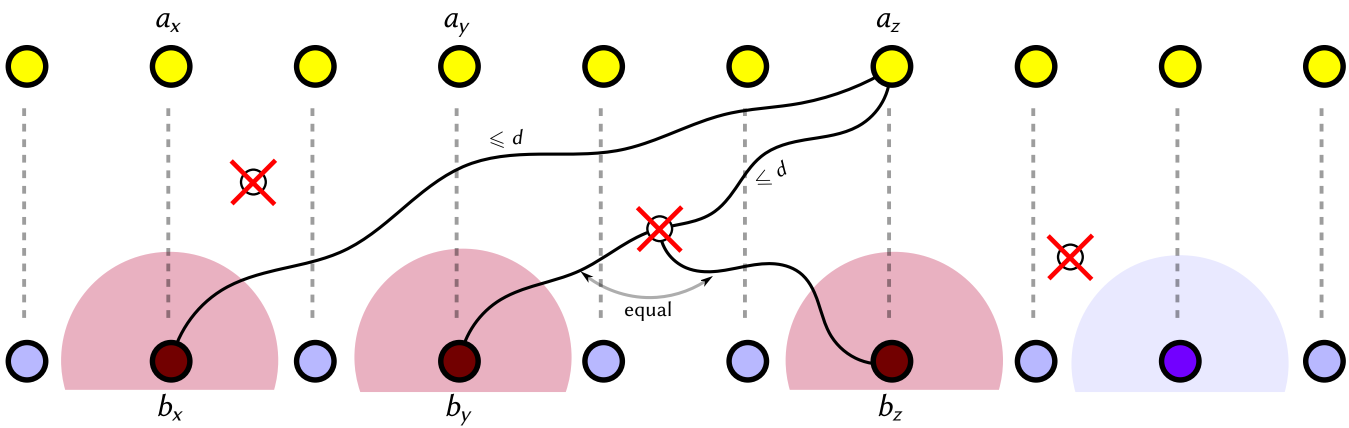
Cor: Maximum semi-ladder order is $\ell := N_{2d}(2(d+1)^{s_{2d}})$.



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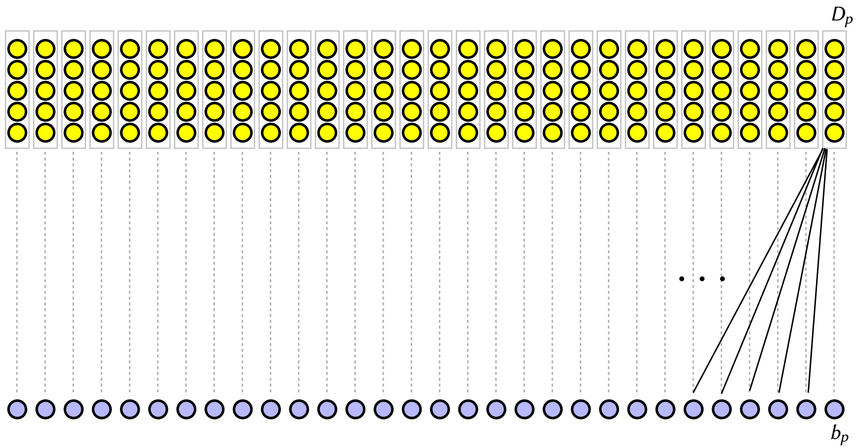

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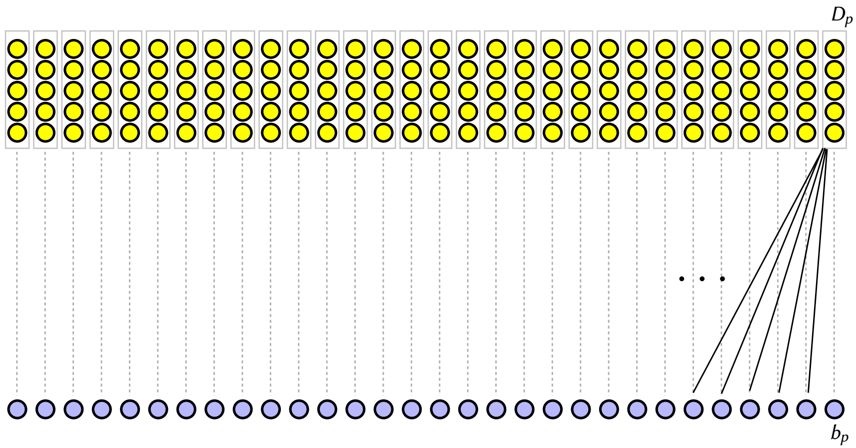

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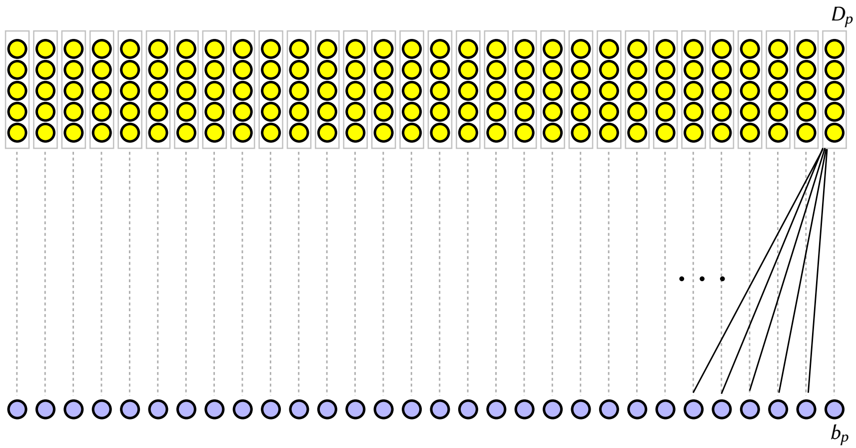

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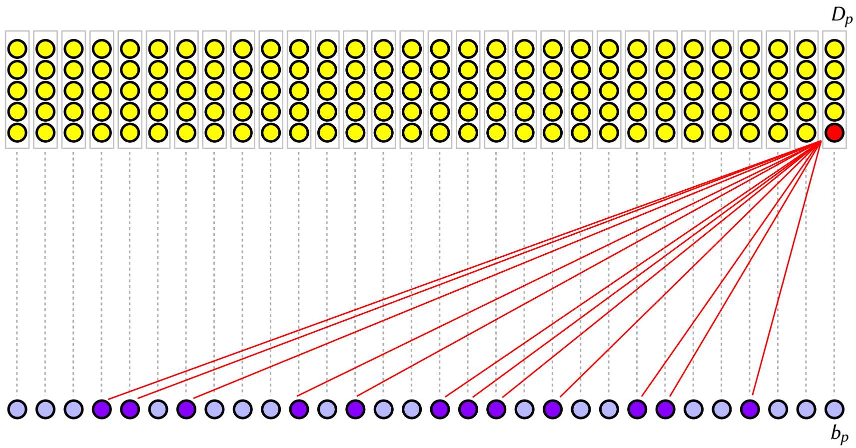

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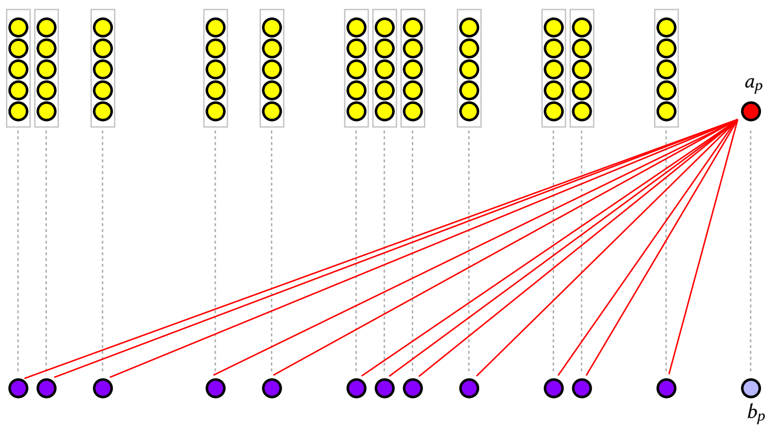

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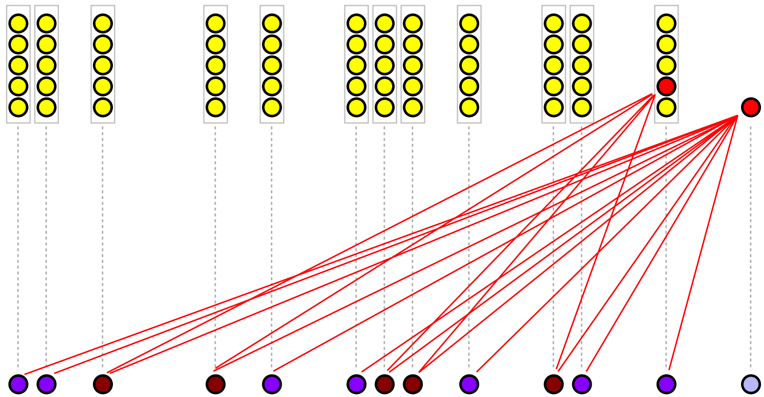

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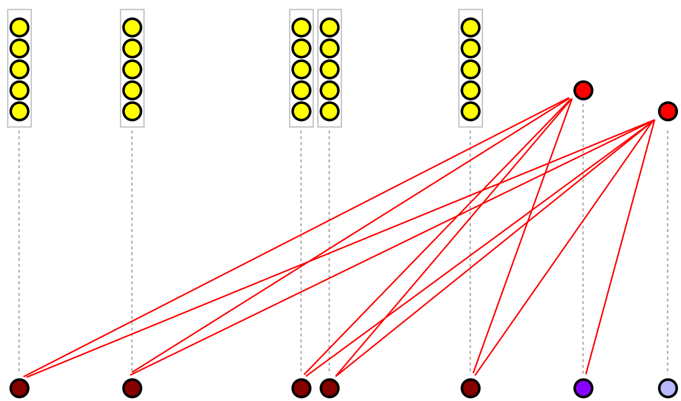

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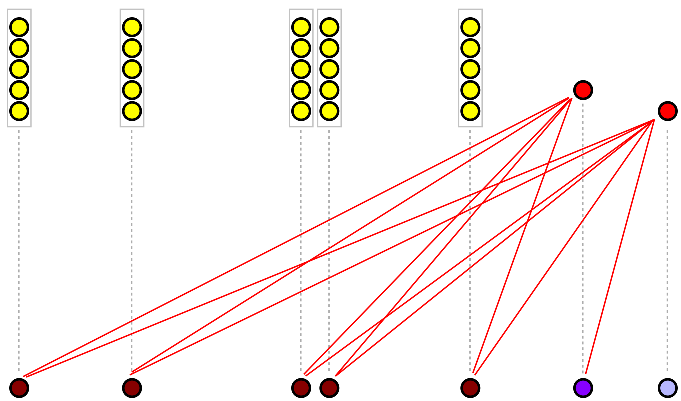

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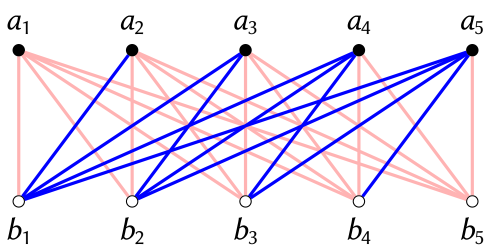

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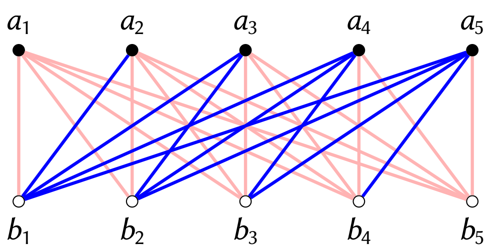

φ-ladder $\leftrightsquigarrow$ **linear order**

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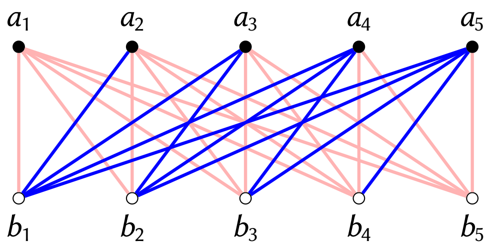

φ-ladder $\rightsquigarrow$ **linear order**

Definition

A class $\mathcal{C}$ is **stable** if for every FO formula $\varphi(x, y)$, there is a **finite** upper bound on the orders of **φ-ladders** in graphs from $\mathcal{C}$.

Beyond Sparsity

Theorem (Adler & Adler; Podewski & Ziegler)

Every **nowhere dense** class is **stable**.

Every **subgraph-closed stable** class is **nowhere dense**.

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Goal: A **theory** of **well-structured dense graphs**.

Further reading

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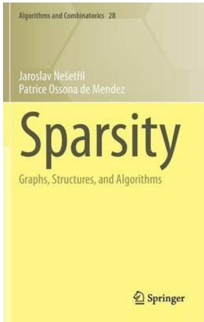
Monograph **Sparsity** of Nešetřil and Ossona de Mendez



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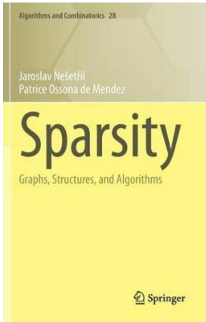

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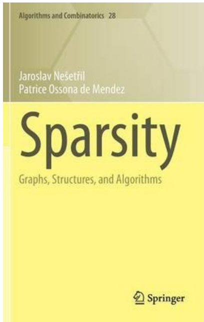

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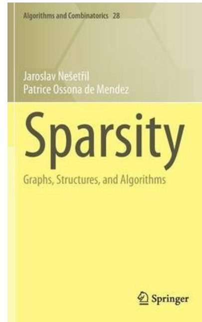

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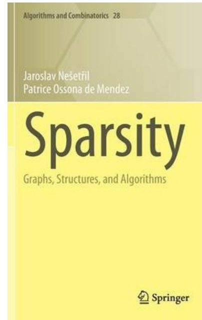

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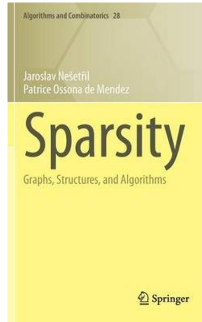

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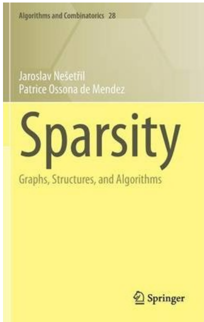

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Thank you for the attention!