Domination Games

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Domination game and its fundamental properties

Rules of the game

- For a graph $G = (V, E)$, the domination number of G is the minimum number, denoted $\gamma(G)$, of vertices in a subset A of V such that $V = N[A] = \cup_{x \in A} N[x]$.

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- If C denotes the set of vertices chosen at some point in a game and $\mathcal{D}$ or $\mathcal{S}$ chooses vertex w, then $N[w] - N[C] \neq \emptyset$.
- $\mathcal{D}$ uses a strategy to end the game in as few moves as possible; $\mathcal{S}$ uses a strategy that will require the most moves before the game ends.

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Introduced in:

Brešar, K., Rall, Domination game and an imagination strategy, SIAM J. Discrete Math. 24 (2010) 979–991.

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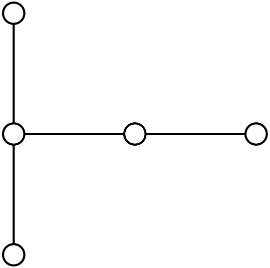
- If $\mathcal{D}$ employs a strategy of selecting vertices from a minimum dominating set A of G, then D-game will have ended when $\mathcal{D}$ has exhausted the vertices from A.

Theorem (Brešar, K., Rall, 2010)

If G is any graph, then $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$. Moreover, for any integer $k \geq 1$ and any $0 \leq r \leq k-1$, there exists a graph G with $\gamma(G) = k$ and $\gamma_g(G) = k + r$.

Game played on a tree



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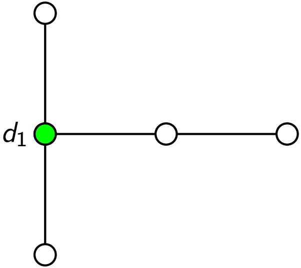

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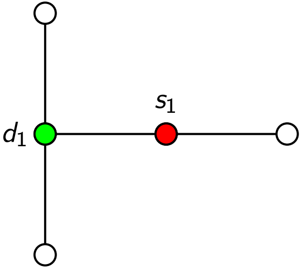

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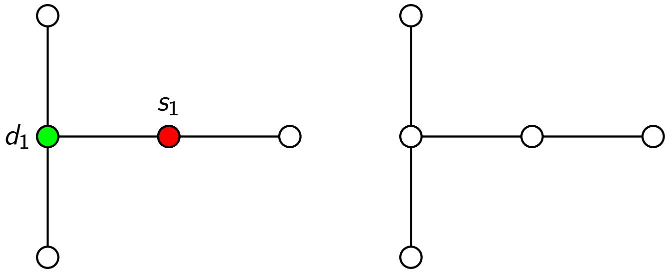

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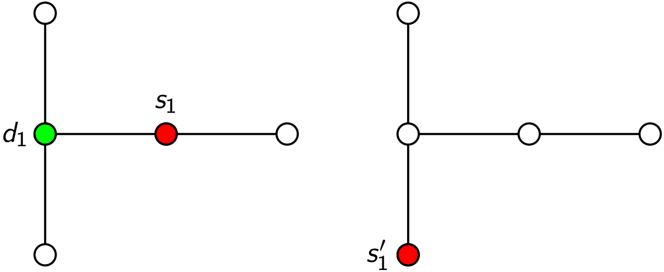

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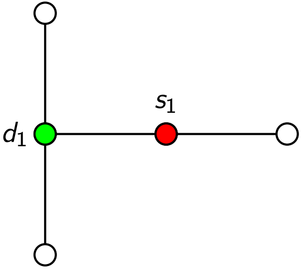

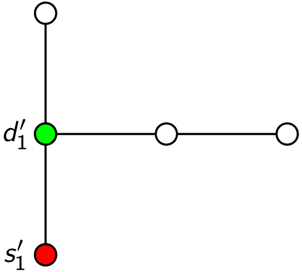

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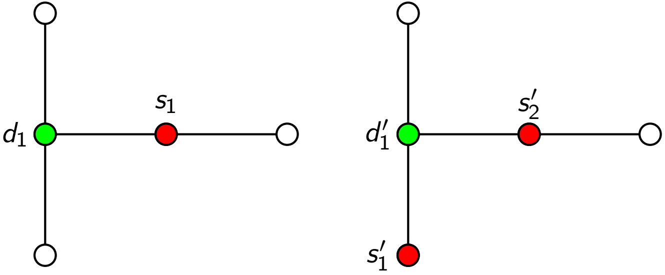

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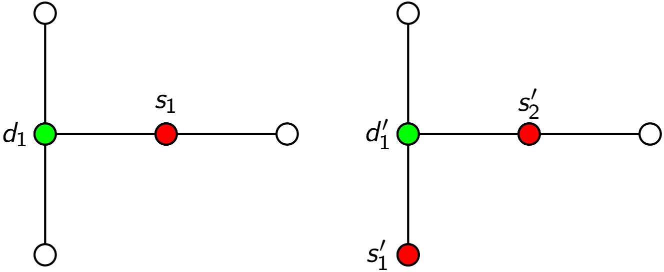
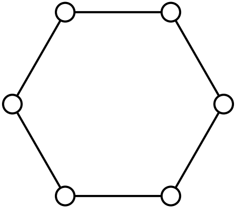


$$\gamma_{\rm g}(T) = 2, \quad \gamma_g'(T) = 3$$

Game played on C_6



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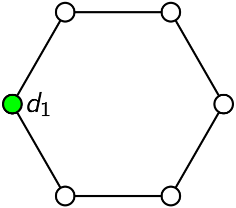

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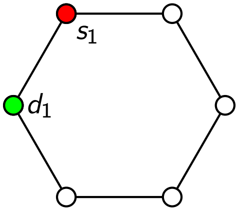

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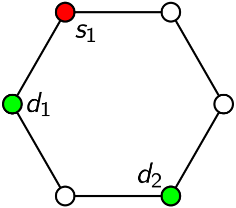

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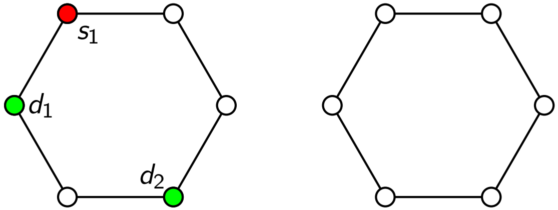

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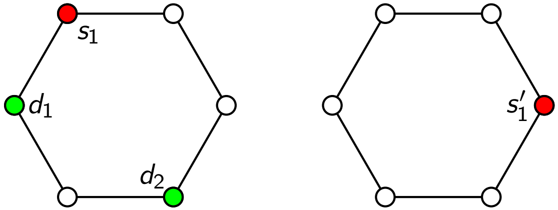

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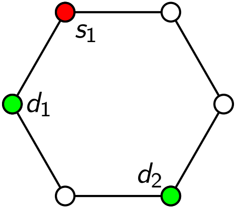

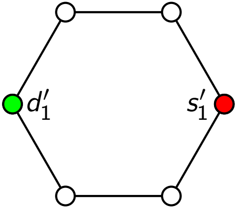

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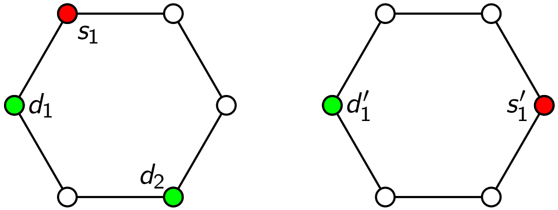

$$\gamma_g(C_6) = 3, \quad \gamma'_g(C_6) = 2$$

γ_g versus γ_g'

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For any graph G, $|\gamma_g(G) - \gamma_g'(G)| \leq 1$.

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Lemma (Kinnersley, West, Zamani, 2013)

(Continuation Principle) Let G be a graph and $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_g(G|A) \le \gamma_g(G|B)$ and $\gamma_g'(G|A) \le \gamma_g'(G|B)$.

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- Suppose Game B is not yet finished. If there are no undominated vertices in Game A, then Game A has finished before Game B and we are done.
- It is $\mathcal{D}$'s move: he selects an optimal move in game B. If it is legal in Game A, he plays it there as well, otherwise he plays any undominated vertex.

Proof of Continuation Principle cont'd

- It is $\mathcal{S}$'s move: she plays in Game A. By the rule, this move is legal in Game B and $\mathcal{D}$ can replicate it in Game B.

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- $\mathcal{D}$ played optimally on Game B. Hence:
  - If $\mathcal{D}$ played first in Game B, the number of moves taken on Game B was at most $\gamma_{\rm g}(G|B)$ (indeed, if $\mathcal{S}$ did not play optimally, it might be strictly less);
  - If $\mathcal{S}$ played first in Game B, the number of moves taken on Game B was at most $\gamma_g'(G|B)$.

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- Hence
  - If $\mathcal{D}$ played first in Game B, then $\gamma_{\rm g}(G|A) \leq \gamma_{\rm g}(G|B)$;
  - If $\mathcal{S}$ played first in Game B, then $\gamma_{\rm g}'(G|A) \leq \gamma_{\rm g}'(G|B)$.

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- By Continuation Principle, $\gamma_g'(G') \leq \gamma_g'(G)$.
- Hence $\gamma_{\rm g}(G) \leq \gamma_g'(G') + 1 \leq \gamma_g'(G) + 1$.

By a parallel argument, $\gamma_g'(G) \leq \gamma_{\rm g}(G) + 1$.

The 3/5-conjecture

Conjecture (Kinnersley, West, Zamani, 2013)

If T is an n-vertex forest without isolated vertices, then

$$\gamma_{\rm g}(T) \le \frac{3n}{5} \quad \text{and} \quad \gamma_g'(T) \le \frac{3n+2}{5}\,.$$

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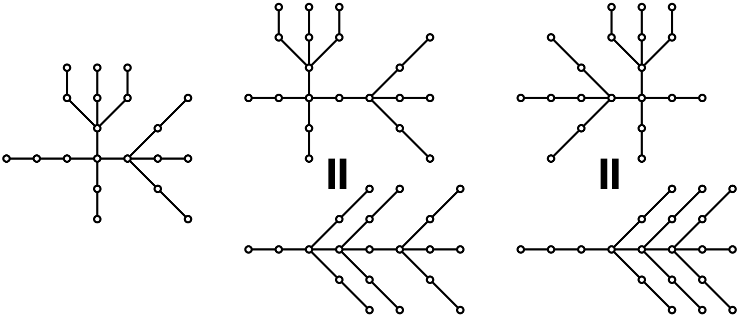
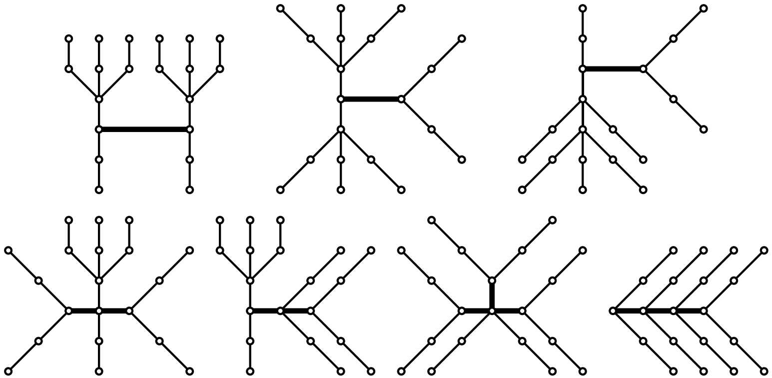
- The conjecture implies the conjecture for forests. But it is NOT clear that the conjectures are equivalent. "*Apply the conjecture for forests to a spanning tree of G*" does not work. A spanning subgraph may have smaller $\gamma_{\rm g}$!

3/5-trees on 20 vertices



3/5-trees on 20 vertices cont'd



Bujtás' approach

- At any moment of the game there are 3 types of vertices:
  - white . . . undominated; of value 3
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  - white . . . undominated; of value 3
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- **Residual graph**: at a given point of the game the graph without red vertices and edges connecting two blue vertices.

Bujtás' approach cont'd

Theorem (Bujtás, 2014)

If T is an n-vertex isolate-free forest with w white vertices and b blue vertices and no two leaves are at distance 4, then

$$\gamma_g(T) \leq \frac{3w + 2b}{5}\,.$$

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Corollary

If T is an n-vertex isolate-free forest with in which no two leaves are at distance 4, then

$$\gamma_g(T) \leq \frac{3n}{5}\,.$$

Partial results

Theorem (Bujtás, 2015)

If G is an n-vertex graphs and $\delta(G) \geq 4$, then

$$\gamma_{\rm g}(G) \leq \frac{15\delta^4 - 28\delta^3 - 129\delta^2 + 354\delta - 216}{45\delta^4 - 195\delta^3 + 174\delta^2 + 174\delta - 216}n\,.$$

Moreover, if $\delta(G) = 3$, then $\gamma_{\rm g}(G) < 0.5574$.

Partial results cont'd

Theorem (Henning, Kinnerlsey, 2016)

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Theorem (Bujtás, 2020)

If G is an isolate-free graph, then $\gamma_{\rm g}(G) \leq \frac{5}{8}n(G)$.

Variety of domination games

Five natural games

In the i^{th} move of the domination game, the choice of v_i is legal if for $v_1, \ldots, v_{i-1}$ chosen so far, the following hold:

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- $N[v_i] \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset$, $v_i \neq v_j$ $(j < i)$ ... L-domination game
- $N[v_i] \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset$ ... LL-domination game

Related invariants

- the game total domination number $\gamma_{\rm tg}(G)$
- the game Z-domination number $\gamma_{\rm Zg}(G)$
- the game L-domination number $\gamma_{\rm Lg}(G)$
- the game LL-domination number $\gamma_{\rm LLg}(G)$

S-game: $\gamma'_{\rm tg}(G)$, $\gamma'_{\rm Zg}(G)$, $\gamma'_{\rm Lg}(G)$, and $\gamma'_{\rm LLg}(G)$

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A motivation for these games:

Theorem (Brešar et al., 2017)

If G is a graph without isolated vertices, then $\gamma_{\rm gr}^{\rm Z} + Z(G) = |V(G)|$. Moreover, the complement of a (minimum) zero forcing set of G is a (maximum) Z-set of G and vice versa.

Hierarchy of the games

Theorem (Brešar et al., 2019)

If G is a graph without isolated vertices, then

- $\gamma_{\rm Zg}(G) \le \gamma_{\rm g}(G) \le \gamma_{\rm Lg}(G) \le \gamma_{\rm LLg}(G)$ *and*
- $\gamma_{\rm Zg}(G) \le \gamma_{\rm tg}(G) \le \gamma_{\rm Lg}(G) \le \gamma_{\rm LLg}(G)$.

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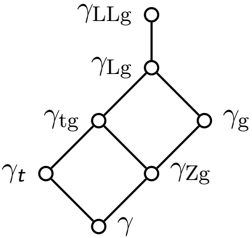
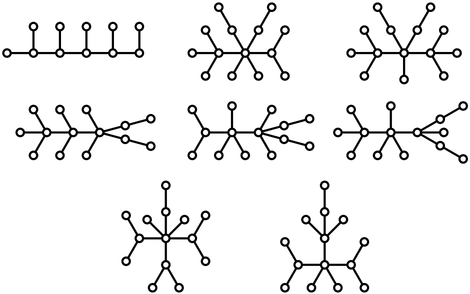


Hierarchy of the games

Smallest trees with pairwise different values:



$$\text{top left tree}: \gamma_{\rm Zg} = 5, \gamma_{\rm g} = 6, \gamma_{\rm tg} = 7, \gamma_{\rm Lg} = 8, \gamma_{\rm LLg} = 9$$
$$\text{top right tree}: \gamma_{\rm Zg} = 5, \gamma_{\rm tg} = 6, \gamma_{\rm g} = 7, \gamma_{\rm Lg} = 8, \gamma_{\rm LLg} = 9$$

Continuation principle

Theorem (Henning, K., Rall, 2015; Brešar et al., 2019)

If G is a graph without isolated vertices and $B \subseteq A \subseteq V(G)$, then

- $\gamma_{\rm tg}(G|A) \le \gamma_{\rm tg}(G|B)$ *and* $\gamma'_{\rm tg}(G|A) \le \gamma'_{\rm tg}(G|B)$
- $\gamma_{\rm Zg}(G|A) \le \gamma_{\rm Zg}(G|B)$ *and* $\gamma'_{\rm Zg}(G|A) \le \gamma'_{\rm Zg}(G|B)$
- $\gamma_{\rm Lg}(G|A) \le \gamma_{\rm Lg}(G|B)$ *and* $\gamma'_{\rm Lg}(G|A) \le \gamma'_{\rm Lg}(G|B)$
- $\gamma_{\rm LLg}(G|A) \le \gamma_{\rm LLg}(G|B)$ *and* $\gamma'_{\rm LLg}(G|A) \le \gamma'_{\rm LLg}(G|B)$

3/4-conjecture

Conjecture (Henning, K., Rall, 2017)

If G is a graph in which every component contains at least three vertices, then $\gamma_{\rm tg}(G) \leq \frac{3}{4}n(G)$.

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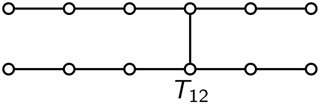
Theorem (Henning, Rall, 2016)

The total domination game 3/4-conjecture is true over the class of graphs G that satisfy both conditions (a) and (b):
(a) *the degree sum of adjacent vertices in G is at least 4 and*
(b) *no two leaves are at distance exactly 4 apart in G.*

$\frac{3}{4}$-conjecture cont'd



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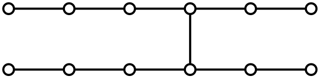

T_{12}

Proposition

If $T \in \{P_4, P_8, T_{12}\}$*, then* $\gamma_{\rm tg}(T) = \gamma'_{\rm tg}(T) = \frac{3}{4}n(T)$.

Critical graphs

G is total domination game critical if $\gamma_{\rm tg}(G) > \gamma_{\rm tg}(G|v)$ holds for every $v \in V(G)$.

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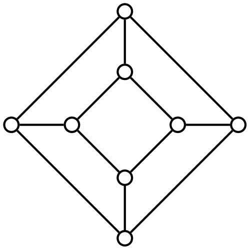
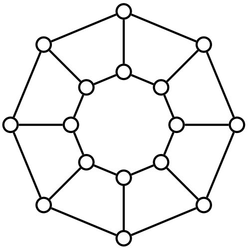
Theorem (Henning, K., Rall, 2018)

If $n \geq 2$, then the path P_n is $\gamma_{\rm tg}$-critical if and only if $n \bmod 6 \in \{2, 4\}$.

Critical graphs cont'd





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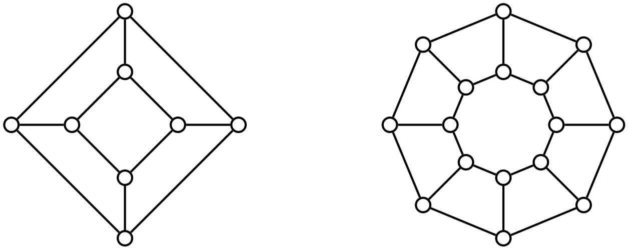
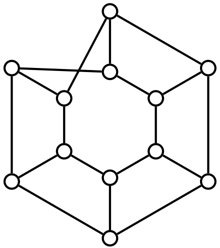
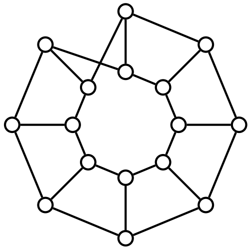


Theorem (Henning, K., 2018)

If $k \geq 1$, then the circular ladder CL_{4k} is $4k$-$\gamma_{\rm tg}$-critical.

Critical graphs cont'd





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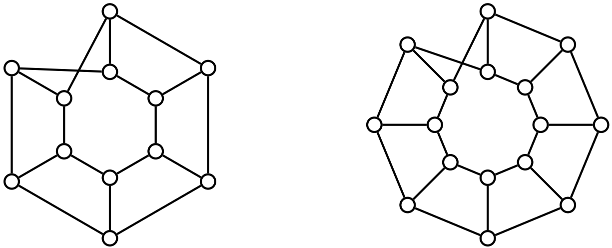

Theorem (Henning, K., 2018)

If $k \geq 1$, then the Möbius ladder ML_{2k} is $2k$-γ_{tg}-critical.

On Z-game

Theorem (Bujtás, Iršič, K., 2020)

If $n \geq 2$ and $\delta(G) \geq 1$, then $\gamma_{\rm tg}(G) = \gamma_{\rm Zg}(G \circ \overline{K}_n)$.

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If G is a weakly claw-free graph, then $\gamma_{\rm Zg}(G) = \gamma_{\rm g}(G)$ and $\gamma'_{\rm Zg}(G) = \gamma'_{\rm g}(G)$.

Other games, open problems, conjectures

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- Maker-Breaker domination game (Duchêne, Gledel, Parreau, Renault, 2020)

Problems and conjectures

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If F is an isolate-free forest satisfying $\gamma_{\rm g}(F) = \frac{3}{5}n(F)$, then every component of F belongs to the family $\mathcal{T}$.

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Problem

Determine the connected graphs G satisfying $\gamma_{\rm tg}(G) = \frac{3}{4}n(G)$.

Problems and conjectures cont'd

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Characterize the $\gamma_{\rm tg}$*-critical trees.*

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Conjecture (Brešar et al., 2019)

If G *is a graph without isolated vertices, then* $\gamma_{\rm Lg}(G) \leq \frac{6}{7}n(G)$.

Thank you for your attention!