8TH PCCC

ALGEBRAIC GRAPH ALGORITHMS

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"EXTENSOR-CODING", STOC 18

CORNELIUS BRAND
HOLGER DELL

OBSERVATIONS

DEF (K-PATH) = SIMPLE PATH OF K VERTICES

DISTINCT VERTICES

$v_1, \dots, v_k \in V$

WITH $v_i v_{i+1} \in E$

DEF (K-PATH PROBLEM)

GIVEN DIGRAPH G, DET. IF G HAS A K-PATH.

DEC. VERSION OF **LONGEST PATH**

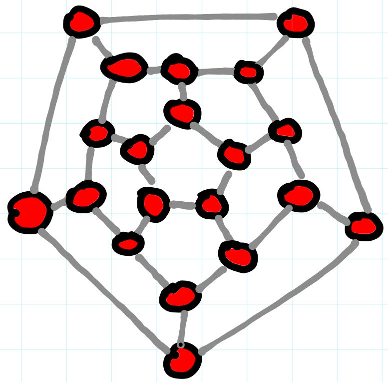
ALG (BRUTE FORCE)

BRUTE-FORCE: $\underbrace{n(n-1) \cdots (n-k+1)}_{k} = O(n^k)$.

COMPLEXITY

PLAUSIBLE $\exp(k, n)$

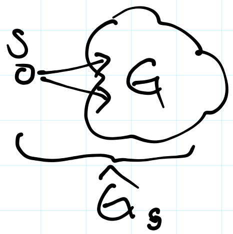
$k := n$ GIVES HAMILTONICITY



FIRST GOAL $\exp(k)\,\mathrm{poly}(n)$

CAN FIX $v_1 = s$

FIX START NODE



G has $\geq k$-path $\Leftrightarrow$ $\hat{G}_s$ has $\geq (k+1)$-path from s

WALK-SUM

DEF (K-WALK)

$v_1,\dots,v_k$ with

$v_iv_{i+1} \in E(G)$. (reps ok).

TODAY: Fix $v_1 = s$.

DEF (WALK-SUM)

some RING $+,\cdot$

no comm.

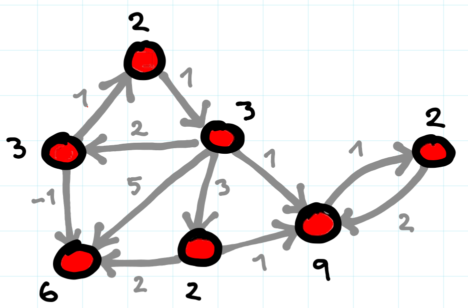
assoc.

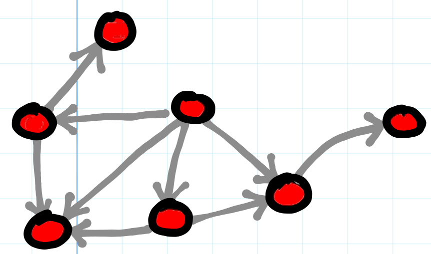
$\alpha : V(G) \cup E(G) \to R$

$$f(G;\alpha) = \sum_{k\text{-walk } v_1\cdots v_k} \alpha(v_1)\alpha(v_1v_2)\cdots\alpha(v_{k-1}v_k)\alpha(v_k).$$

• $\binom{n}{k}$ terms of $2k-1$-products. TIME

EXAMPLE, K=4





$3\cdot1\cdot9\cdot3\cdot2\cdot1 + 3\cdot2\cdot1\cdot2\cdot5.$

+

ALG. FOR WALK-SUM

DP: For $1\le r\le k$, $v\in V(G)$, define

$$S(r,v) = \sum_{\substack{r\text{-walks } v_1\cdots v_k \text{ w.} \\ v_k = v}} f(v_1,\dots,v_k).$$

Then

$$S(r,v) = \sum_{uv\in E(G)} S(r-1,u)\cdot w(uv)\cdot w(v) \quad \forall v\in V$$

$O(k(n+m))$. MULTIPLICATIONS.

K ROUNDS OF MEMORYLESS BFS

COMPUTE

$$[1 \ \cdots \ 1]\,[A(G)]^{k}\begin{bmatrix} w(v_1)\\ \vdots \\ w(v_n)\end{bmatrix}$$

ALG FOR K-PATH IN DAGS

Set $\alpha(v)=\alpha(e)=1 \ \forall v\in V, e\in E$

Return "yes" iff $f(G;\alpha)\neq 0$

WARM-UP: RANDOM PERMUTATION

ALG FOR K-PATH

REPEAT $k!$ TIMES

$\pi =$ top. ord. of $V(G)$.

$E(G') := \{ uv \mid \pi(u) \le \pi(v) \}$

$LP_{DAG}(G')$

ANALYSIS

$Pr(\text{FALSE POS}) = 0$

$Pr(\text{FALSE NEG}) \le 1 - Pr\left(\overset{\pi(1) < \pi(2) < \cdots < \pi(k)}{① \to ② \to \cdots \to ⓚ} \right)$

$= 1 - \frac{\overset{\pi(k+1)}{n} \cdot \overset{\pi(k+2)}{(n-1)} \cdots \overset{\pi(n)}{(n-k+1)} \cdot \overset{\pi(1)<\cdots<\pi(k)}{1}}{n!} = 1 - \frac{1}{k!}$

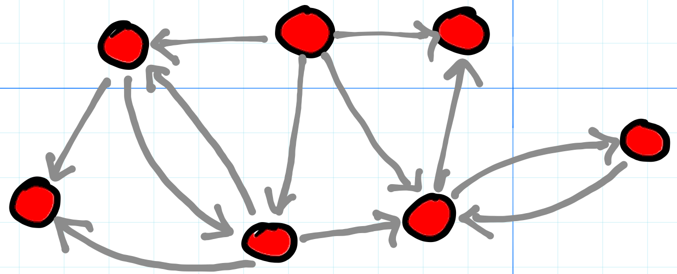
$Pr(\text{SUCCESS}) \ge \frac{1}{2}$ "FPT"

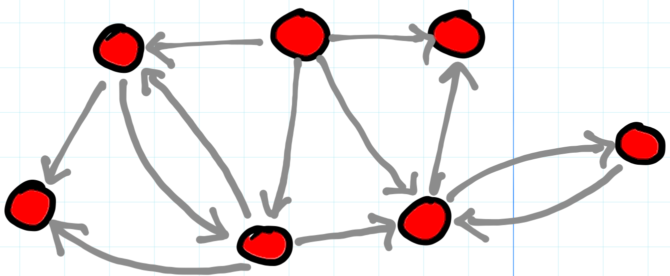
TIME : $O(k!(n+m)) = f(k)\,\text{poly}(n)$

RANDOMIZED

NEXT: GET TO $\exp(k)\,\text{poly}(n)$.

EXAMPLE





COLOUR-CODING

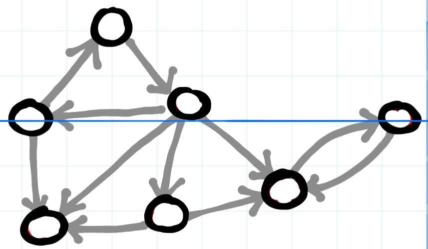
ALON-YUSTER-ZWICK

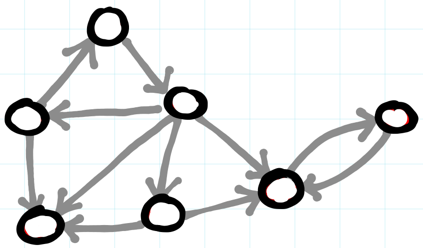
DEF

$\chi : V \to \{1,2,\dots,k\}$

AT RANDOM

EXAMPLE





DYNAMIC PROGRAMMING

$$C(R,v) = \begin{cases} 1 & \text{if } \exists P = v_1,\dots,v_{|R|} \\ & \chi(V(P)) = R \\ 0 & \text{o.w.} \end{cases}$$

P USES EXACTLY COLS FROM R

G has k-PATH $\iff \bigvee_{u \in V} C(K,u)$

$C(\emptyset, v) = 0.$

$$C(R,v) = \bigvee_{uv \in E(G)} C(R-\chi(v), u).$$

ANALYSIS

P is RAINBOW BY χ

$$\Pr[\text{FALSE NEG}] = 1 - \frac{k!}{k^k} \sim 1 - e^{-k}.$$

ALL COLS.

TIME & SPACE $O(2^k k(n+m))$

WEDGE PRODUCT

FIELD F $(=\mathbb{Q})$, $k \in \mathbb{N}$.

F^k VECTOR SPACE $+, \cdot$

Basis $e_1, e_2, \dots, e_k$ for F^k

$y = e_1 + 2e_3$ $\quad a \in F^k$ written as $a_1e_1 + \dots + a_ke_k = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}$

VECTORS

$x + y \in F^k$. $\quad c \in F, x \in F^k$: $cx \in F^k$

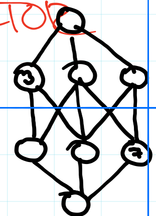
ΛF^k, EXTERIOR ALGEBRA

$x = 3e_{\{1,2\}} + 7e_{\{3\}}$ $\quad y = e_{\{1\}} + 2e_{\{3\}}$ $\quad$ EXTENSOR

Basis e_I for $I \subseteq \underbrace{\{1, \dots, k\}}_{k}$

VECTOR

$x \in \Lambda F^k$ written as $\sum_{I \subseteq k} a_I e_I$, $x =$



EXTENSORS

- $F^k \subseteq \Lambda F^k$: $e_i \cong e_{\{i\}}$.
- $F = \Lambda^0 F^k$

WEDGE PRODUCT $\wedge$

- $\wedge$ $\quad (x \wedge y) \wedge z = x \wedge (y \wedge z)$

$(x + y) \wedge z = x \wedge z + yz$

$x \wedge y := -y \wedge x$, $x, y \in F^k$

$\Rightarrow$ $x \wedge x = -x \wedge x$

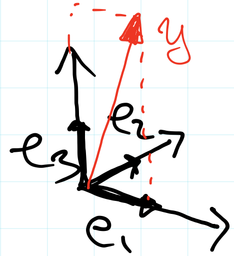
$e_7 \wedge e_5 = e_{\{5,7\}}$

$e_5 \wedge e_7 = -e_7 \wedge e_5$

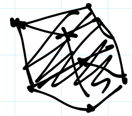
$e_5 \wedge e_5 = 0$

$$e_I \wedge e_J = \begin{cases} 0 & \text{if } I \cap J \neq \emptyset \\ (-1)^{\dots} e_{I \cup J} & \text{if } I \cap J = \emptyset \end{cases}$$

GEOMETRY



$e_2 \wedge e_1 =$  $e_1 \wedge e_2$ 

$e_1 \wedge e_2 \wedge e_3 =$ 

EX: $x \wedge y = (3e_1 \wedge e_2 + 7e_3) \wedge (e_1 + 2e_3)$

$= 6e_1 \wedge e_2 \wedge e_3 + 7e_1 \wedge e_3$

$x \wedge x = 0$

$y \wedge y = (3e_1 \wedge e_2 + 7e_3) \wedge (3e_1 \wedge e_2 + 7e_3)$

$= 21\, e_1 \wedge e_2 \wedge e_3 + 21\, e_1 \wedge e_2 \wedge e_3$

$x_1, x_2, x_3 \mapsto x_1 \wedge x_2 \wedge x_3$, $\quad x_i \in F^k$

$= \det \begin{bmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{bmatrix} \cdot e_1 \wedge e_2 \wedge e_3$

$x \in \Lambda F^k$, $y = F^k$

$$x \wedge y = \sum_{I \cap J \neq \emptyset} (-1)^{\dots} x_I y_J e_{I \cup J}$$

CONCRETE DEF.

WEDGE PRODUCT

ABSTRACT DEF.

F RING $(F=\mathbb{Q})$. F^k VECTOR SPACE

$\dim(F^k)=k$, $F^k=\langle e_1,e_2,\dots,e_k\rangle$

VECTOR SPACE WITH MULT.

EXTERIOR ALGEBRA ΛF^k

$\dim(\Lambda F^k)=2^k$

$x\wedge y=-y\wedge x$, $x,y\in F^k$

$x\wedge x=-x\wedge x=0$

ΛF^k AS A VECTOR SPACE:

BASIS $\langle e_I\rangle_{I\subseteq\overbrace{\{1,2,\dots,k\}}^{k}}$

$x\in\Lambda F^k$ WRITTEN AS $x=\sum_{I\subseteq K}\alpha_I e_I$

EXTENSOR

EX: $x=3e_{\{1,2\}}+7e_{\{3\}}$

$y=e_{\{1\}}+2e_{\{3\}}\simeq e_1+2e_3\in F^k$

$F^k\subseteq\Lambda F^k$: $e_i\hat{=}e_{\{i\}}$

$+$: $2x+y=e_{\{1\}}+9e_{\{3\}}+6e_{\{1,2\}}$.

$\wedge$: $x\wedge y=3e_{\{1,2\}}\wedge e_{\{1\}}+6e_{\{1,2\}}\wedge e_{\{3\}}+7e_{\{3\}}\wedge e_{\{1\}}+14e_{\{3\}}\wedge e_{\{3\}}$

$e_i\wedge e_j=0$ if $i=j$

$e_i\wedge e_j=-e_j\wedge e_i$

$e_{\{1\}}\wedge e_{\{2\}}=e_{\{1,2\}}$

$e_{\{2\}}\wedge e_{\{1\}}=-e_{\{1,2\}}$

$e_3\wedge e_5\wedge e_4\wedge e_1=\quad e_{\{1,3,4,5\}}$

DEF:

$$e_I\wedge e_J=\begin{cases}0 & \text{if } I\cap J\neq\emptyset\\ \underbrace{(-1)^{\#\mathrm{inv}(I,J)}}_{\text{sign}}e_{I\cup J}\end{cases}$$

$I=\{i_1,\dots,i_r\}$, $J=\{j_1,\dots,j_s\}$

VECTOR SPACE $F^k=\langle e_1,\dots,e_k\rangle$

e_i = canonical basis vector $e_i=\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}_i$

$F^k\ni\sum_i\lambda_i e_i$, $\lambda_i\in F$

"FREE"

TENSOR ALGEBRA $T(F^k)=\bigoplus_{r=0}^{\infty}(F^k)^{\otimes r}$

$e_1\otimes e_2\otimes e_1\otimes e_1-1e_3\otimes e_1+3e_1\otimes e_3\in T(\mathbb{Q}^3)$

BASIS $e_{i_1}\cdots e_{i_r}$ FOR EACH SEQ. $i_1,\dots,i_r=:\underline{i}$

NONCOMMUTATIVE POLYNOMIALS $X_1\cdot X_2\cdot X_1\cdot X_1-1X_3X_1+3X_1X_3$

$\dim(T(F^k))=\infty$

IDEAL $\mathcal{J}=\{v\cdot v\mid v\in F^k\}\subseteq T(F^k)$

$I\subseteq A$ IDEAL: LINEAR SUBSPACE S.T.

$i\in I\Rightarrow a\cdot i\cdot b\in I$

A/I CONGRUENCE CLASS, VECTOR SPACE.

$[a]=\{a+i\mid i\in I\}$.

$\Lambda F^k=:$ QUOTIENT ALGEBRA $T(F^k)/I$

REPRESENTATION & COMPUTATION

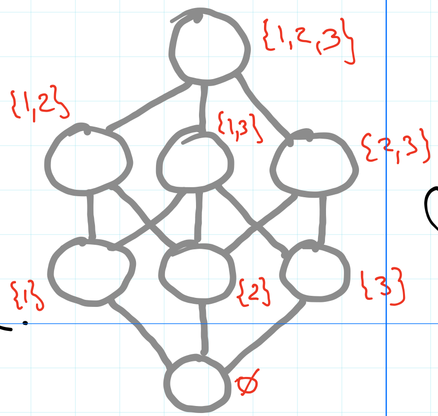
REPRESENT EXTENSOR

STORE

$$x = \sum_{I \subseteq K} a_I e_I$$

AS COEFFICIENTS $\{a_I\}_{I \subseteq K}$.

2^k SPACE



$$x = 7 + 3e_{\{1\}} + \tfrac{1}{2} e_{\{1,2\}} + 7e_{\{1,3\}} + 5e_{\{1,2,3\}}$$

CONNECTION TO DETERMINANT

$x_1 \cdots x_k \in F^k$

$$(x_1, x_2, \dots, x_k) \mapsto x_1 \wedge x_2 \wedge \cdots \wedge x_k$$

$(a_1 e_1 + \cdots + a_{2k} e_k)$

$$= \boxed{}\; e_1 \wedge e_2 \wedge \cdots \wedge e_k$$

$$\det \begin{bmatrix} | & & | \\ x_1 & \cdots & x_k \\ | & & | \end{bmatrix}$$

ALTERNATING, MULTILINEAR

$$(e_1, e_2, \dots, e_k) \mapsto 1 \cdot e_1 \wedge e_2 \wedge \cdots \wedge e_k$$

EXTENSOR ∧ VECTOR

$$x \wedge y = \underbrace{\Big(\sum_{I \subseteq K} a_I e_I\Big)}_{2^k \text{ terms}} \wedge \underbrace{\Big(\sum_{i \in K} a_i e_i\Big)}_{k \text{ terms}}$$

$2^k k$ TIME 2^k SPACE

EXTENSOR-CODING

DEF (EXTENSOR-CODING)

$\xi : V(G) \to \wedge(\mathbb{F}^k)$, $W = w_1 \cdots w_k$

DEF (WALK EXTENSOR)

$$\xi(W) = \xi(w_1) \wedge \xi(w_2) \wedge \cdots \wedge \xi(w_k)$$

MAIN LEMMA

<u>LEMMA</u> (MAIN): If W is not a path & $\xi(v) \in \mathbb{F}^k$

$$\xi(W) = 0.$$

In particular

$$\underbrace{f(G;\xi)}_{\text{WALK-SUM}} = \sum_{W \text{ walk}} \xi(W) = \sum_{P \text{ paths}} \xi(P).$$

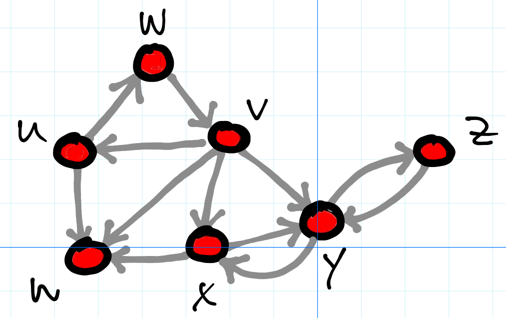
<u>Pf</u> $W = \cdots v \cdots v \cdots$

$$\xi(W) = \cdots \xi(v) \wedge \cdots \wedge \xi(v) \wedge \cdots$$

$$= (-1)^{\circledast} \cdots \xi(v) \wedge \xi(v) \cdots$$

$$= 0. \quad \square$$

EXAMPLE



$v \wedge u \wedge w \wedge v \quad = 0$ ☺

$w \wedge v \wedge y \wedge x + w \wedge v \wedge y \wedge x = 0$ ☹

$w \wedge v \wedge x \wedge w = 0$ ☹

ISSUES

ISSUES: ① $\xi(P) = 0$?

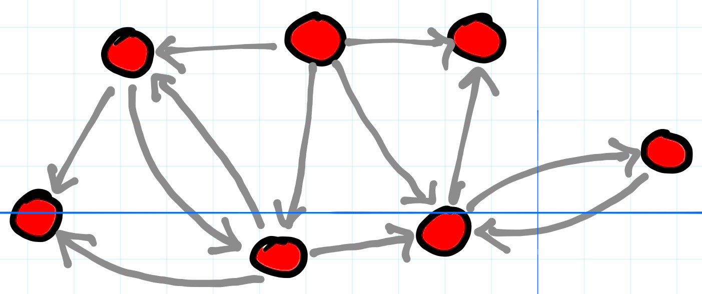
② $\xi(P) + \xi(Q) = 0$?

COLOUR-CODING REVISITED

1ST ATTEMPT

$$\chi : V(G) \to \Lambda(\mathbb{F}^k)$$

$$v \mapsto e_i \,, \quad i \in_{\text{RANDOM}} \{1, \ldots, k\}.$$



$$\chi(P) = e_{r_1} \wedge e_{r_2} \wedge \cdots \wedge e_{r_k} = \begin{cases} 0 \\ \pm 1 \end{cases}$$

$$\Pr(\tilde{\chi}(P) = 1) = \frac{\overset{\text{ALL DIFFERENT}}{k!}}{\underset{\text{CHOICES OF } e_{r_1} \cdots e_{r_k}}{k^k}} \sim e^k.$$

2ND ATTEMPT

$$\bar{\chi} : V(G) \to \Lambda(\mathbb{F}^{2k})$$

$$v \mapsto \begin{pmatrix} e_i \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ e_i \end{pmatrix}$$

$$\bar{\chi}(P) = \begin{pmatrix} e_{r_1} \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ e_{r_1} \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} e_{r_k} \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ e_{r_k} \end{pmatrix}$$

$$= \det \left[\begin{array}{ccc|ccc} e_{r_1} & e_{r_2} \cdots & e_{r_k} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & e_{r_1} & e_{r_2} & e_{r_k} \end{array} \right] = \left(\det \left[e_{r_1} \cdots e_{r_k} \right] \right)^2 \in \{0, 1\}$$

ALGORITHM

ALG: Repeat e^k times:

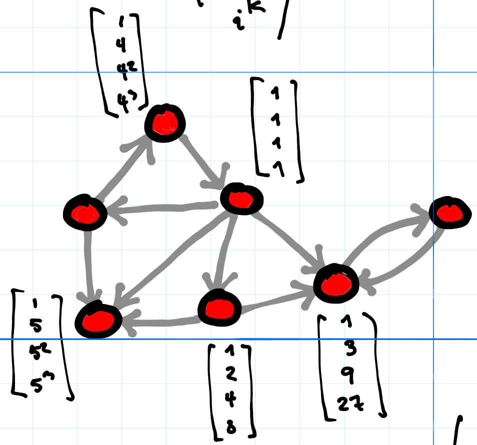
- Choose $\bar{\chi} : V(G) \to \underbrace{\begin{pmatrix} e_i \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ e_i \end{pmatrix}}_{\text{A ZEON}}$
- Compute $[f(G; \bar{\chi}) \neq 0]$

VANDERMONDE VECTORS

VANDERMONDE

$$\phi : V(G) \to \mathbb{Q}^k$$

$$v_i \mapsto \begin{pmatrix} i^0 \\ i^1 \\ i^2 \\ \vdots \\ i^k \end{pmatrix}.$$



$$\phi\left(\overset{i_1}{\circ} \to \overset{i_2}{\circ} \to \cdots \overset{i_{k-1}}{\circ} \to \overset{i_k}{\circ} \right) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ i_1 & i_2 & & i_k \\ i_1^2 & i_2^2 & \cdots & \\ \vdots & & & \\ i_1^{k-1} & i_2^{k-1} & & i_k^{k-1} \end{vmatrix}$$

$$= \prod_{\substack{i_a, i_b \\ a<b}} (i_a - i_b) \neq 0. \quad \text{always.}$$

ALG (UNAMBIGUOUS K-PATH)

- $\phi(v_i) := (i^0, \dots, i^{k-1})^T \quad \forall i$
- return $[f(G;\phi) \neq 0]$

DETERMINISTIC, $2^k \mathrm{poly}(n)$ TIME.

ALG (DET. K-PATH)

$$\bar\phi : V(G) \to \Lambda \mathbb{Q}^{2k}$$

$$v_i \mapsto \begin{pmatrix} \phi(v_i) \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ \phi(v_i) \end{pmatrix} = \begin{bmatrix} i^0 \\ i^1 \\ \vdots \\ i^{k-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ \vdots \\ 0 \\ i^0 \\ i^1 \\ \vdots \\ i^{k-1} \end{bmatrix}$$

$$\bar\phi(w_1 \cdots w_k) = \begin{vmatrix} \ast & 0 \\ 0 & \underbrace{\ast}_{k} \end{vmatrix} \Big\} k \cdot (-1)^{\Omega \leftarrow \binom{k}{2}}$$

DETERMINISTIC, TIME $4^k \mathrm{poly}(n)$.

VERY FAR FROM $2.5961^k \mathrm{poly}(n)$ [Meirav Zehavi 2015].

EDGE CODING

"ALGEBRAIC FINGERPRINTING"

KOUTIS, WILLIAMS (2008-9) using $\mathbb{F}[\mathbb{Z}_2^k] \simeq \Lambda(\mathbb{F}^k)$ — $\mathbb{F}$ has char. 2.

DEF

$\rho : V(G) \cup E(G) \to \mathbb{Q}^k$.

$v_i \mapsto \phi(i) = [i^0, i^1, \dots, i^{k-1}]^T$

$uv \mapsto r_{uv}$, randomly $\{1, \dots, 2(k-1)\}$

ALG

Set up ρ as above

Return $[f(G;\rho) \neq 0]$

POLYNOMIAL IDENTITY TESTING

Introduce X_{uv} $\forall uv \in E(G)$

Consider

$$g(G) = \sum_{\substack{\text{paths} \\ v_1 \cdots v_k}} \underbrace{\phi(v_1 \cdots v_k)}_{\det[\phi(v_1) \cdots \phi(v_k)] \cdot e_{\{1,\dots,k\}}} \cdot X_{v_1v_2} X_{v_2v_3} \cdots X_{v_{k-1}v_k}$$

Polynomial of deg $k-1$ in m variables

$[e_1 \wedge \dots \wedge e_k] g \in \Lambda^k \mathbb{F}^k[X_{uv}]$

P uniquely det'd by its edges, so $g(G) \neq 0$. — zero polynomial

$f(G;\rho) = g(G)_{X_{uv} \mapsto r_{uv}}$

at $X_{uv} \mapsto r_{uv}$, so

$$\Pr(f(G;\rho) \neq 0) \leq \frac{k-1}{2(k-1)} = \frac{1}{2} \quad \blacksquare$$

TIME : $2^k \text{poly}(n)$.

SUMMARY

EXTENSOR-CODING UNIFIES & GENERALISES SEVERAL K-PATH ALGORITHMS

- $v_i \mapsto (i^0, i^1, \dots, i^{k-1})$ DET. $4^k \mathrm{poly}(n)$, $2^k \mathrm{poly}(n)$ if $|\mathcal{P}| \leq 1$

- $e_j \mapsto 17$ RANDOM INT ALGEBRAIC FINGERPRINTING

- $v_i \mapsto \begin{pmatrix} +1 \\ -1 \\ -1 \\ +1 \end{pmatrix}$ RANDOM BERNOULLI COUNT k-PATHS IN $4^k \mathrm{poly}(n)$

- $v_i \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ RANDOM BASIS VECTOR COLOUR-CODING

ALSO MIMICKS REPRESENTATIVE SETS. [BRAND, DELL, H. "EXTENSOR-CODING" STOC '18]

OPEN: COMPLEXITY OF $x \wedge y$ if $x, y \in \Lambda \mathbb{F}^k$.

KNOWN: $O(2^{k\omega/2})$.