A useful tool in combinatorics: Intersecting set-pair systems

Z. Füredi
(newest results with A. Gyárfás and Z. Király)

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Extended Abstract

A *cross intersecting set pair system* (SPS) of size m:

$$(\{A_i\}_{i=1}^m, \{B_i\}_{i=1}^m) \text{ with } A_i \cap B_i = \emptyset \text{ and } A_i \cap B_j \neq \emptyset.$$

It is an important tool of extremal combinatorics. Bollobás' classical result states that $m \leq \binom{a+b}{a}$ if $|A_i| \leq a$ and $|B_i| \leq b$ for each i.

Our central **problem** is to see how this bound changes with additional conditions (**proofs, applications and generalizations**).

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1-cross intersecting set pair system: $|A_i \cap B_j| = 1$ for all $i \neq j$.

We show connections to perfect graphs, clique partitions of graphs, and finite geometries. The max size of a 1-cross intersecting SPS is

- at least $5^{n/2}$ for n even, $a = b = n$,
- equal to $\left(\lfloor \frac{n}{2} \rfloor + 1\right)\left(\lceil \frac{n}{2} \rceil + 1\right)$ if $a = 2$ and $b = n \geq 4$,
- at most $|\cup_{i=1}^m A_i|$,
- asymptotically n^2 if $\{A_i\}$ is a linear hypergraph ($|A_i \cap A_j| \leq 1$ for $i \neq j$). ,

Some standard notation

$[n] := \{1, 2, \ldots, n\}$

$\binom{S}{k} :=$ set of k-sets, 2^S power set

$\deg_G(x)$ degree of vertex x of graph $G = (V, E)$

$N_G(x) \subset V$, neighborhood

$T \subseteq V(\mathcal{H})$ is a cover (transversal) of the hypergraph $\mathcal{H} = (V, \mathcal{E})$
if $T \cap e \neq \emptyset \quad \forall e \in \mathcal{E}$.

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$\mathcal{M} \subseteq \mathcal{E}$ is a matching (parallel edges) if $e \cap e' = \emptyset \; \forall e \neq e' \in \mathcal{M}$.
$\nu(\mathcal{H}) := \min |\mathcal{M}|$. (matching number of $\mathcal{H}$).

$\nu = 1 \iff \mathcal{H}$ is intersecting hpgr.

Cross intersecting set pair systems

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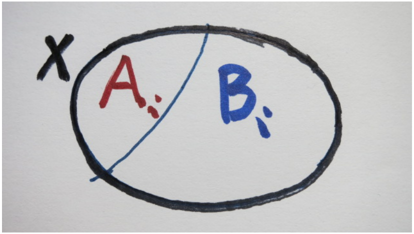
Theorem (Bollobás 1965)

$(A_1, , B_1), (A_2, B_2), \ldots, (A_m, B_m)$ set pairs with
$|A_i| \leq a$, $|B_i| \leq b$ and
$A_i \cap B_i = \emptyset$, and
$A_i \cap B_j \neq \emptyset$ for all $i \neq j$, then $m \leq \binom{a+b}{a}$.

Best possible:
$\mathcal{A} := \binom{X}{a}$, $\mathcal{B} := \binom{X}{b}$,



Proof of Bollobás' theorem

Conj'd by Berge/ Ehrenfeucht, Mycielsky 1973. Proof by Jaeger, Payan 1971, Tarjan 1975, Katona 1974, (Alon / Frankl).

Given $(\mathcal{A}, \mathcal{B})$, a cross intersecting SPS. $X := \cup\mathcal{A} \cup \mathcal{B}$.

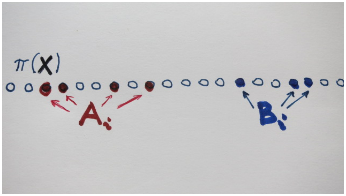


Figure: A type i permutation of X.

Call a permutation type i if $A_i <_\pi B_i$.

Note that type i $\neq$ type j.

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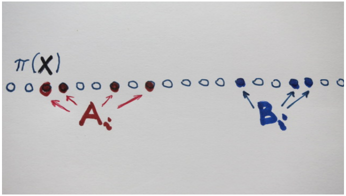

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Note that type $i \neq$ type j.

$\mathbb{P}\text{rob}(\pi \text{ is of type } i) = 1/\binom{|A_i|+B_i|}{|A_i|}$.

Therefore

$$\sum_i \frac{1}{\binom{|A_i|+B_i|}{|A_i|}} \leq 1. \quad \square$$

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2.
To investigate 1-cross intersecting SPS. $(|A_i \cap B_j| = 1)$
New results are joint with Gyárfás and Király, and related to one of my favorite structures, finite affine and projective planes

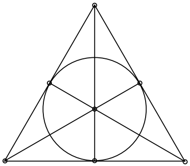

Figure: The Fano plane.

The kernel of intersecting families

DEF: Suppose $\mathcal{H}$ is intersecting. S is a *kernel* of $\mathcal{H}$ if

$$S \cap e \cap f \neq \emptyset \quad \forall e, f \in \mathcal{H}.$$

E.g., the kernel of $\{\binom{S}{r}\}$ (for $|S| < 2r$) is itself,
the kernel of $K_{1,m}$ (star) is a singleton.

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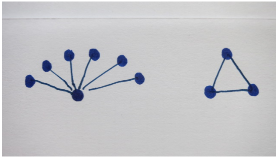
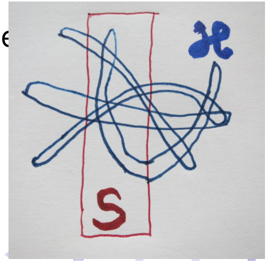
Theorem (Całczyńska-Karłowicz 1964)

$\forall r\ \exists CK(r) < \infty$ *such that:*
If $\mathcal{H}$ is intersecting, $|e| \leq r\ \forall e \in \mathcal{H}$, then $\exists S$, $|S| \leq CK(r)$.

$CK(1) = 1$, $CK(2) = 3$, $CK(3) \geq 7$,
$CK(q+1) \geq q^2 + q + 1$ if $\exists$ projective plane





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Remove element x if $\mathcal{H}|(X \setminus \{x\})$ is still intersecting.
Repeat until we get a critical S:

$$\forall x \in S \;\; \exists A(x), B(x) \in \mathcal{H}_{new} \quad A(x) \cap B(x) = \{x\}.$$

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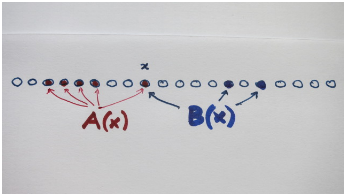

Figure: A type x permutation of X.

Call a permutation type x if $A(x) \leq_\pi B(x)$. Type $x \neq$ Type y.
$\mathbb{P}\mathrm{rob}(\pi$ is of type $x) = \ldots,\;\; \sum_{x \in S} \mathbb{P}\mathrm{rob} \leq 1,\;\; CK(r) \leq \frac{r}{2}\binom{2r-1}{r}$.

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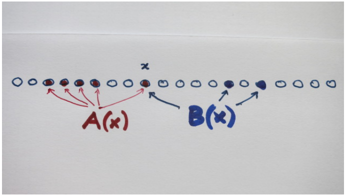

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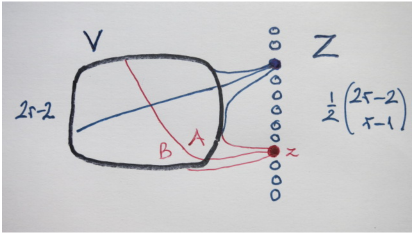
New results, see: Kang, Ni, Shan 2017, Polcyn, Ruciński 2017, Henning, Yeo 2014, Tuza 1994/1996 (surveys), ...

The kernel could be exponential

A **construction** by Erdős, Lovász 1973.



Take $|V| = 2r - 2$, $|Z| = \frac{1}{2}\binom{2r-2}{r-1}$, $V \cap Z = \emptyset$.
$\forall$ partition Π of V, $|A| = |B| = r - 1$ assign $z := z(\Pi) \in Z$.

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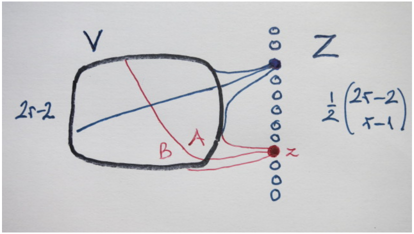

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$\forall$ partition Π of V, $|A| = |B| = r - 1$ assign $z := z(\Pi) \in Z$.
Let $\{A(z), B(z)\} := \{A \cup \{z\}, B \cup \{z\}\}$.
Then $\mathcal{H}$ is intersecting, critical, $V(\mathcal{H})$ can not be shrunk

$$CK(r) \geq (2r - 2) + \frac{1}{2}\binom{2r-2}{r-1} \quad \square$$

New results, generalizations, see: Alon, Füredi 1987, Talbot 2004, Tuza 1994/1996 (surveys), ...

Sperner families, LYM inequality

DEF: $\mathcal{H} \subset 2^{[n]}$ is a **Sperner** family if $A \not\subset B \ \forall A, B \in \mathcal{H}$.

Theorem (Sperner)

$$|\mathcal{H}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

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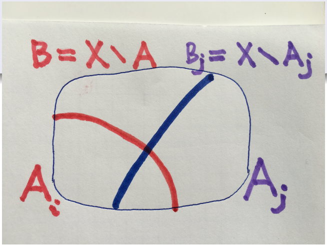

Theorem (LYM: Lubell, Yamamoto, Meshalkin)

$\sum_{A \in \mathcal{H}} \frac{1}{\binom{n}{|A|}} \leq 1.$

Proof: $(A, X \setminus A)$ is an cross intersetcing SPS. □
Equality iff $|A| = r \ \ \forall A \in \mathcal{H}$.

Cross intersecting k-tuples

DEF: $(A_i, B_i, \ldots, Z_i)_{1 \le i \le m}$ form a family of cross intersecting k-tuples if these k sets are pairwise disjoint and

$$\forall i \neq j \quad \exists X \neq Y \in \{A, B, \ldots, Z\} \text{ such that } X_i \cap Y_j \neq \emptyset.$$

Theorem (Tuza)

If $p_1 + \cdots + p_k = 1, \forall p_j > 0$, *then*

$$\sum_i p_1^{|A_i|} \cdots p_k^{|Z_i|} \le 1$$

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Corollary (Take $p_1 = p_2 = 1/2$.)

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$\{A_i, B_i\}_{1 \le i \le m}$ *cross intersecting SPS then* $m \le 2^{\max\{|A_i|+|B_i|\}}$.

Even more, it is enough to suppose that for all $i \neq j$

$$\max\{|A_i \cap B_j|, |A_j \cap B_i|\} > 0.$$

Other results on k-tuples: Alon 1985.

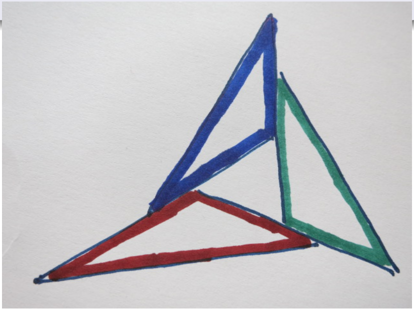
A geometric application, tangent simplices

DEF: $S, S' \subset \mathbb{R}^d$ are **tangent** simplices if

$$\mathrm{int}\, S \cap \mathrm{int}\, S' = \emptyset \text{ and } \dim(S \cap S') = d - 1.$$

Theorem (Perles)

Suppose $S_1, \ldots, S_m \subset \mathbb{R}^d$ pairwise tangent.
Then $m \leq 2^{d+1}$.

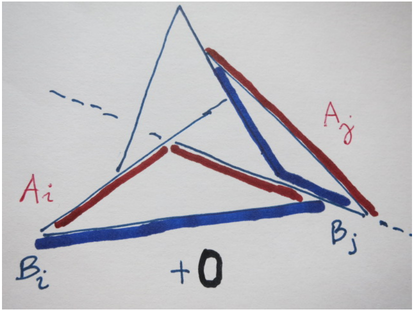


Four tangent triangles

J. Zaks: $h(2) = 4$, $h(3) = 8$, $h(d) \geq 2^d$.

Tangent simplices, a proof using SPS

Perles' proof.
Suppose $S_1, \ldots, S_m \subset \mathbb{R}^d$ pairwise tangent.
Let $H_1, \ldots, H_v$ all hyperplanes, tangent to some faces
$(v \leq m(d+1))$.



Take a general point O and given S_i
$A_i :=$ tangent hyperplanes H_α such that $O \in H_\alpha^+$ and $S_i \subset H_\alpha^+$.
Then $\{|A_i \cap B_j|, |A_j \cap B_i|\} = \{0, 1\}$. □

Pairs of disjoint subspaces

DEF: $A, B \subset \mathbb{R}^d$ are **disjoint** linear subspaces if

$$A \cap B = \{\mathbf{0}\}, \quad \text{i.e., } \dim(A+B) = \dim(A) + \dim(B).$$

Non-disjoint: $\dim(A \cap B) \geq 1$.

Theorem (Lovász)

Suppose that $(A_i, B_i)_{1 \leq i \leq m}$ is a cross intersecting family of disjoint pairs of subspaces of dimensions a and b, i.e.,

$$\dim(A_i) \leq a, \dim(B_i) \leq b \text{ and}$$

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Implies Bollobás.

Given a cross intersecting SPS $(\mathcal{A}, \mathcal{B})$: take $d = |\cup \mathcal{A} \cup \mathcal{B}|$ pairwise orthogonal vectors $\{\mathbf{e}_t : t \in [d]\}$ and let $\widehat{A}_i := \mathrm{Span}(\{\mathbf{e}_t : t \in A_i\})$.

A consequence of Lovász geometric method

Theorem (ZF cross t-intersecting families)

$(A_1, , B_1)$,...,(A_m, B_m) *set pairs with* $|A_i| \le a$, $|B_i| \le b$ *and*

$$A_i \cap B_i \le t, \text{ and } A_i \cap B_j > t \text{ for all } i \ne j.$$

Then $m \le \binom{a+b-2t}{a-t}$.

Best possible:
$V = X \cup T$, $|X| = a + b - 2t$, $|T| = t$ and
$\mathcal{A} := \{A \in \binom{V}{a}, T \subset A\}$, $\mathcal{B} := \{B \in \binom{V}{b}, T \subset B\}$,

Another consequence, skew cross intersecting SPS's

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Alon and Kalai: used skew cross intersecting SPS's
to prove the Upper Bound Theorem of McMullen:
An exact formula for the max number of t dim faces of an
n-vertex convex polytope with n vertices.

τ-critical graphs

DEF: (Gallai) The graph G is τ-critical if

$$\tau(G \setminus e) < \tau(G) \quad \forall e \in E(G)$$

E.g., $C_{2\tau-1}$, $K_{\tau+1}$.

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G is τ-critical, then $|V(G)| \leq 2\tau$.

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Proof of the EHM thm:
Consider $A(e) := e$ for edge $e \in E(G)$, and
$B(e) :=$ a covering set of the edges of $E(G) \setminus \{e\}$.
They form a cross intersecting SPS of set sizes 2 and $\tau - 1$. □

τ-critical hypergraphs

$$\tau(\mathcal{H} \setminus e) < \tau(\mathcal{H}) \quad \forall e \in E(\mathcal{H})$$

E.g., Erdős-Lovász construction for $CK(r)$, $K_r^{\tau+r-1}$.

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Theorem (Bollobás 1965)

$\mathcal{H}$ is τ-critical, r-uniform, then $|E(\mathcal{H})| \le \binom{\tau+r-1}{r}$.

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What about $f(\tau, r) := \max |V(\mathcal{H})|$, $\mathcal{H}$ is τ-critical, r-uniform.

τ-critical hypergraphs

$$\tau(\mathcal{H} \setminus e) < \tau(\mathcal{H}) \quad \forall e \in E(\mathcal{H})$$

E.g., Erdős-Lovász construction for $CK(r)$, $K_r^{\tau+r-1}$.

Theorem (Bollobás 1965)

$\mathcal{H}$ is τ-critical, r-uniform, then $|E(\mathcal{H})| \le \binom{\tau+r-1}{r}$.

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Theorem (Petruska, Szemerédi $r = 3$, Gyárfás, Lehel, Tuza 1982, finally Tuza)

$\mathcal{H}$ is τ-critical, then $|V(\mathcal{H})| \le \binom{\tau+r-1}{r-1} + \binom{\tau+r-2}{r-2}$.

Best upper bound is still unknown.

PART II. 1-cross intersecting families

DEF: A *1-cross intersecting set pair system* (SPS) of size m:

$(\{A_i\}_{i=1}^m, \{B_i\}_{i=1}^m)$ with $A_i \cap B_i = \emptyset$ and $\boxed{|A_i \cap B_j| = 1}$ $\forall i \neq j$.

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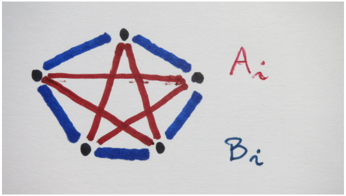
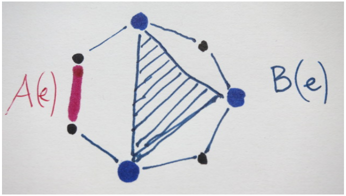




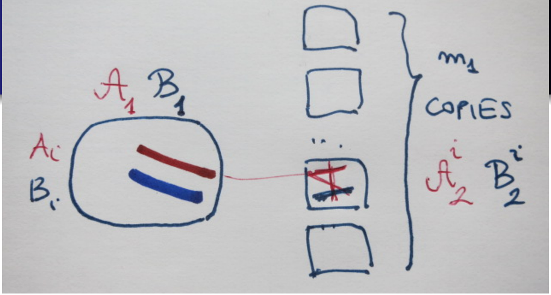
A C_5 and $\overline{C_5}$ form a $(2, 2)$-bounded 1-cross int' SPS.

It is optimal: $m_2(*, *, 1) = 5$

The edges of a C_{2n+1} and independent covers:

a $(2, n)$-bounded 1-cross intersecting SPS.

The product construction



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If (a_1, b_1)-bounded and (a_2, b_2)-bounded 1-cross intersecting SPS exist with sizes m_1 and m_2, then $(a_1 + a_2, b_1 + b_2)$-bounded 1-cross intersecting SPS also exists of size $m_1 \cdot m_2$.

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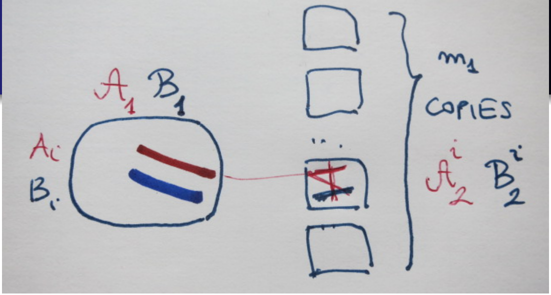

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Take m_1 pairwise disjoint ground sets $V_2^1, \dots, V_2^{m_1}$ with copies $(\mathcal{A}_2^i, \mathcal{B}_2^i)$, e.g., $\mathcal{A}_2^i = \{A_{i,1}, \dots, A_{i,m_2}\} \dots$

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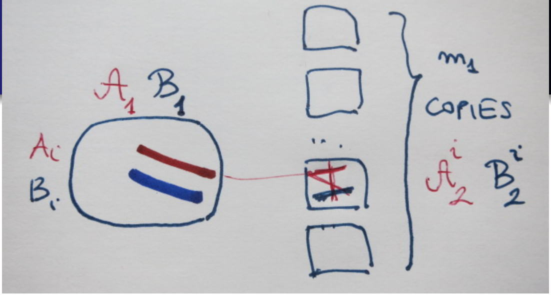

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The pairs $A'_{i,j} = A_i \cup A_{i,j}$, $B'_{i,j} = B_i \cup B_{i,j}$ form a 1-cross intersecting SPS, $|A'_{i,j}| \le a_1 + a_2$ and $|B'_{i,j}| \le b_1 + b_2$. □

1-cross intersecting SPS can be exponential

Corollary

There exists an (n, n)-bounded 1-cross intersecting SPS of size $5^{n/2}$ if n is even and of size $2 \cdot 5^{(n-1)/2}$ if n is odd. □

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We have another example, the pairs $(\{i, i{+}1, i{+}2\}, \{i{+}3, i{+}6, i{+}9\})$ taken $\pmod{10}$ has 10 vertices.

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Samuel Spiro (sspiro@ucsd.edu) informed us that his computer program successfully checked that 10 is indeed the largest size,
$m_1(*, *, 1) = 2$,
$m_2(*, *, 1) = 5$,
$m_3(*, *, 1) = 10$,
$m_4(*, *, 1) \geq 25$.

Limit exists

Fekete's Lemma on subadditive sequences implies

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A challenging **problem** is to decrease essentially Bollobás' upper bound:

Conjecture

There exists a positive ε *such that* $m_n(*,*,1) \leq (1-\varepsilon)\binom{2n}{n}$ *for every* $n \geq 2$.

A Fischer's inequality for cross 1-intersecring SPS

Proposition ($m \le |\cup \mathcal{A}|$)

Let $(\mathcal{A}, \mathcal{B})$ be 1-cross intersecting, $V := \cup \mathcal{A}$. The char. vectors of the edges of $\mathcal{A}$ are linearly independent in $\mathbb{R}^V$.

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Let $\mathbf{a}_i$ (resp. $\mathbf{b}_i$) denote the characteristic vector of A_i (resp. B_i), $\mathbf{a}_i(v) = 1$ **for** $v \in V$ if and only if $v \in A_i$. Otherwise $\mathbf{a}_i(v) = 0$.

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Take the dot product with $\mathbf{b}_j$. Since $|A_i \cap B_j| = 1$ for $i \neq j$ and $|A_j \cap B_j| = 0$, we get

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Add up for all j: $(m-1) \sum_{i=1}^{m} \lambda_i = 0$, consequently $\sum_{i=1}^{m} \lambda_i = 0$ and thus $\lambda_j = 0$ for all j.

Lovász' characterization of perfect graphs

A special case of the previous Proposition can be used in the non-trivial part of Gasparyan's 1996 proof of Lovász's theorem:

A graph G is perfect if and only if

$$|V(H)| \leq \alpha(H)\omega(H) \qquad (1)$$

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Proof. In a minimal imperfect graph G there is a 1-cross intersecting SPS of size $m = \alpha(G)\omega(G) + 1$ defined by independent sets and complete subgraphs. By the previous Proposition, $|V(G)| \geq \alpha(G)\omega(G) + 1$, contradicting (1). □

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Corollary (Lovász)

A graph is perfect if and only if its complement is perfect.

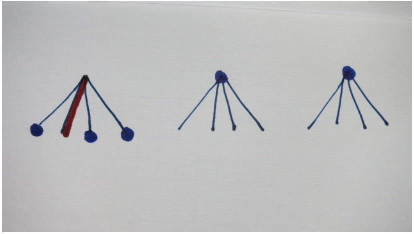
$(2, n)$-*bounded* 1-cross intersecting SPS

Theorem (The case of $\mathcal{A}$ is a graph.)

Let $n \geq 4$, and let $(\mathcal{A}, \mathcal{B})$ be a $(2, n)$-bounded 1-cross intersecting SPS of size m. Then

$$m \leq \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(\left\lceil \frac{n}{2} \right\rceil + 1\right).$$

Best possible. For $n = 2, 3$ the exact values are $m = 5, 7$.



An extremal family: $\left\lfloor \frac{n}{2} \right\rfloor + 1$ copies of stars with $\left\lceil \frac{n}{2} \right\rceil + 1$ edges.

Linear hypergraphs and cross 1-intersecting SPS

A hypergraph $\mathcal{H}$ is called *linear* if the intersection of any two different edges has at most one vertex.
E.g., affine planes $AG(2,q)$. Usually $V(AG(2,q)) = \mathbb{F}_q^2$, q^2 vertices and the hyperedges = $q^2 + q$ lines.

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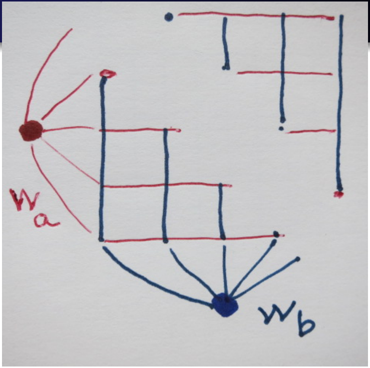
If one of $(\mathcal{A}, \mathcal{B})$, say $\mathcal{A}$, in an SPS is linear, then (without any assumption on $|B_i \cap B_j|, |A_i \cap B_j|$).

Proposition (26)

The size of an (n, n)-bounded cross intersecting SPS $(\mathcal{A}, \mathcal{B})$ with linear $\mathcal{A}$ is at most $n^2 + n + 1$. □

Double stars



The vertex set of a *double star of size s* consist of $\{v_{i,j} \mid 1 \leq i,\ j \leq s,\ i \neq j\}$ and two additional vertices w_a and w_b.
Define for $i \in [s]$ $A_i := \{w_a\} \cup \{v_{i,j} \mid 1 \leq j \leq s,\ j \neq i\}$ and

$$B_i := \{w_b\} \cup \{v_{j,i} \mid 1 \leq j \leq s,\ j \neq i\}.$$

$(\mathcal{A}, \mathcal{B})$ is a 1-cross intersecting SPS of size s containing s-element sets such that both $\mathcal{A}$ and $\mathcal{B}$ are 1-intersecting.

Notation and general setting

Let $a, b > 0$ and $I_A, I_B, I_{\text{cross}}$ three sets of non-negative integers.
Let $m(a, b, I_A, I_B, I_{\text{cross}})$ the maximum size m of a cross intersecting SPS $(\mathcal{A}, \mathcal{B})$ with the following conditions.

i) $A_i \cap B_i = \emptyset$ for every $1 \leq i \leq m$,

ii-iii) $|A_i| \leq a$, $|B_i| \leq b$ for every $1 \leq i \leq m$,

iv-v) $|A_i \cap A_j| \in I_A$, $|B_i \cap B_j| \in I_B$ for every $1 \leq i \neq j \leq m$,

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We suppose that $0 \notin I_{\rm cross}$, and $m \geq 2$.
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Bollobás' theorem:

$$m(a, b, *, *, *) = \binom{a+b}{a},$$

More notations

Our Theorem on p.25 states (for $n \geq 4$)

$$m(2, n, *, *, 1) = \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \left(\left\lceil \frac{n}{2} \right\rceil + 1 \right).$$

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E.g., Proposition p. 26:

$$m_n(01\text{-int}, *, *) \leq n^2 + n + 1.$$

The origin of the new problems, The *thickness* of a clique partitions

Given a graph G, a clique (Biclique) partition of $E(G)$
= parts are complete (complete bipartite) graphs.
The *thickness* of a partition is the maximum s such that every vertex $x \in V(G)$ appears in at most s cliques (bicliques).

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The maximum m such that B_{2m} has a biclique partition of thickness n is $m_n(*, *, 1)$.

The case of both $\mathcal{A}$ and $\mathcal{B}$ are linear hypergraphs

Then the bound of Prop. (26) can be approximately halved.

Theorem (a bit complicated double counting)

Suppose that $(\mathcal{A}, \mathcal{B})$ is an (n, n)-bounded 1-cross intersecting SPS of size m such that both $\mathcal{A}$ and $\mathcal{B}$ are linear hypergraphs. Then $m \le \frac{1}{2}n^2 + n + 1$. I.e., $m_n(1\text{-int}, 1\text{-int}, 1) \le \frac{1}{2}n^2 + n + 1$.

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Bellow we give three constructive lower bounds, large cross intersecting SPS such that $\mathcal{A}$ is an intersecting linear hypergraph, showing that our results asymptotically the best possible, i.e.,

$m_n(1\text{-int}, *, 1)$ and $\boxed{m_n(1\text{-int}, 1\text{-int}, *) = n^2 - o(n^2)}$,

and $m_n(1\text{-int}, 1\text{-int}, 1) = \frac{1}{2}n^2 - o(n^2)$.

(We only give details of the boxed statement.)

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The function $m_n(I_A, I_B, I_{\mathrm{cross}})$ is monotone increasing in n,
we need only for a dense set of special values of n.

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We need $(\mathcal{A}, \mathcal{B})$ such that $|A_i| = n$, $|B_i| = n$ for every $i \in m$, both $\mathcal{A}$ and $\mathcal{B}$ are 1-intersecting of sizes $m_n = n^2 - o(n^2)$.

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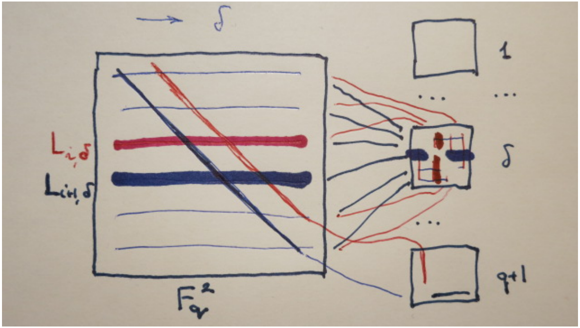
$A_i \cap B_i = \emptyset \ \forall i \in [m]$, and
$A_i \cap B_j \neq \emptyset$ for every $1 \leq i \neq j \leq m$.

Second step: The product of an affine plane and the double star

$AG(2, q)$ has
$q + 1$ directions
(parallel classes).
Each class has
q lines.
Let δ be a
direction.
$L_{1,\delta}, \ldots, L_{q,\delta}$ the
lines of this class.



Take $q + 1$ copies of the double star of size q.

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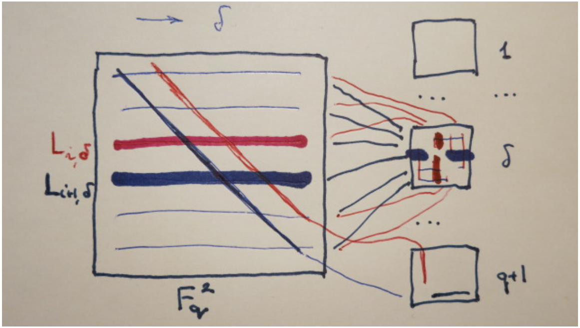

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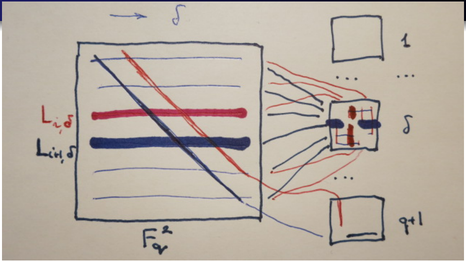
Let $A_{i,\delta} := L_{i,\delta} \cup A_i^\delta$ and $B_{i,\delta} := L_{i+1,\delta} \cup B_i^\delta$.

We obtain $m_{2q} \geq q^2 + q$. Call this construction $\mathcal{H}(2q)$.

Cross intersecting almost projective planes
Third step.



Suppose $p + 2q \leq n < p + 4q$, (p, q primes) where
$$p \leq q^2 + q \text{ and } n - O(n^{5/8}) < p < n - 2\sqrt{n}.$$

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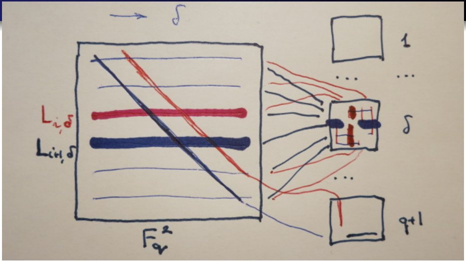

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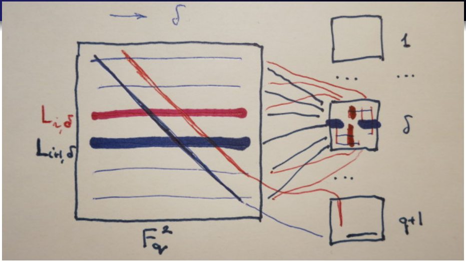

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The size of $\mathcal{H}(2q)$ is $q^2 + q \geq p$ thus we need only the first p set pairs from it. □

The End

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THANKS FOR YOUR ATTENTION!