Erdős–Ko–Rado: Structure & Sparsification



8^{th} Polish Combinatorial Conference

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Today's plan

I. The Classics
   - a (re-)introduction to the Erdős–Ko–Rado Theorem

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I. The Classics
- a (re-)introduction to the Erdős–Ko–Rado Theorem

II. Spectral Stability
- a robust stability statement

I. The Classics

Chapter outline

1. Intersecting Families

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2. The Erdős–Ko–Rado Theorem

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2. The Erdős–Ko–Rado Theorem

3. The Hoffman Bound

Chapter outline

1. Intersecting Families
2. The Erdős–Ko–Rado Theorem
3. The Hoffman Bound
4. Proving Erdős–Ko–Rado

§1 Intersecting Families

I. The Classics

Erdős–Ko–Rado: Structure & Sparsification

A very real real-world example

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  - Adopted worldwide $\rightarrow$ herd immunity
  - Social distancing a thing of the past

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- Form a guest list such that:
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Objectives

- Form a guest list such that:
  - Every pair has common interests to discuss
  - Invite as many friends as possible

A tale of three friends

A tale of three friends

Billiards
Football
Maths
Physics

Billiards
Computers
Ice cream
Movies
Maths

Darts
Movies
Trebuchets

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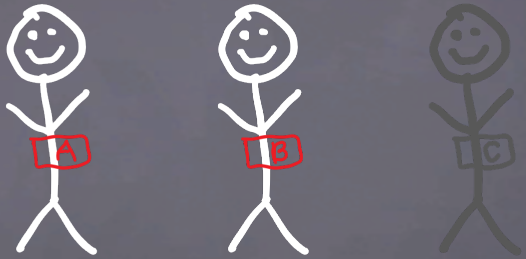

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A mathematical formulation

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  - Universal set of possible interests
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Definition (Intersecting family)

A family $\mathcal{F} \subseteq 2^{[n]}$ is intersecting if, for all $F, F' \in \mathcal{F}$, we have $F \cap F' \neq \emptyset$. That is, $\mathcal{F}$ does not contain a disjoint pair.

Large parties

An extremal problem

How large can an intersecting family $\mathcal{F} \subseteq 2^{[n]}$ be?

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Constructions

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- # maximum intersecting families is $2^{\Theta\left(2^n/\sqrt{n}\right)}$

§2 The Erdős–Ko–Rado Theorem

I. The Classics

Erdős–Ko–Rado: Structure & Sparsification

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A restricted setting

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Can we do better?

The Erdős–Ko–Rado Theorem

Theorem (Erdős–Ko–Rado, 1961)

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For all $n \geq 2k$, if $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Moreover, if $n > 2k$, then the only maximum intersecting families are the stars.

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- Triangle-intersecting graph families
- ...

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The antisocial network

Definition (Kneser graph)

Given $1 \leq k \leq n$, the Kneser graph $KG(n,k)$ has vertices $V = \binom{[n]}{k}$, and edges $E = \{\{S,T\} : S \cap T = \emptyset\}$.

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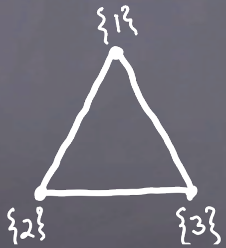

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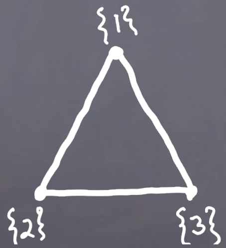

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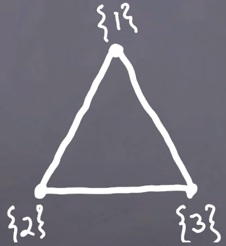

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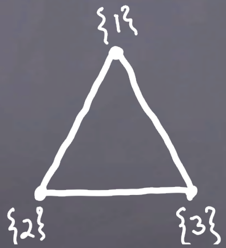

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"All graphs are ugly, except the Petersen graph." — Martin Aigner

Features of the Kneser graph

Symmetry

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- Cliques in $KG(n,k) \leftrightarrow k$-uniform hypergraph matchings

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Theorem (Erdős–Ko–Rado, 1961)

For all $n \geq 2k$, $\alpha\left(KG(n, k)\right) = \binom{n-1}{k-1}$.

§3 The Hoffman Bound

I. The Classics

Erdős–Ko–Rado: Structure & Sparsification

Adjacency matrices

Definition (Adjacency matrix)

Given an n-vertex graph $G = (V, E)$, its adjacency matrix $A_G \in \{0, 1\}^{V \times V}$ is defined by

$$A_G(u, v) = \begin{cases} 1 & \text{if } \{u, v\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

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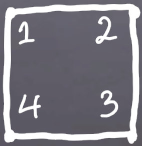

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

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Eigenvalues

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$\Rightarrow A_G\vec{1} = d \cdot \vec{1}$ $\square$

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Notation

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Let G be a connected d-regular graph on n vertices. Then the following hold.

(a) $d \geq \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n \geq -d$.

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Given an n-vertex graph G with adjacency matrix A_G, we denote its eigenvalues by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

Observation

We have $\sum_{i=1}^{n} \lambda_i = \mathrm{tr}(A_G) = 0$. *In particular,* $\lambda_1 \geq 0 \geq \lambda_n$.

Lemma

Let G be a connected d-regular graph on n vertices. Then the following hold.

(a) $d \geq \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n \geq -d$.

(b) $\lambda_n = -d$ *if and only if G is bipartite.*

Counting edges

Observation

Let G be a graph, $U \subseteq V$, and let $\vec{f}$ be the characteristic vector of U. That is,

$$\vec{f}(v) = \begin{cases} 1 & \textit{if } v \in U, \\ 0 & \textit{otherwise.} \end{cases}$$

Then $\vec{f}^T A_G \vec{f} = 2e(U)$.

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$$\vec{f}^T A_G \vec{f}$$

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$$= \sum_{u \in U} |\{v \in U : \{u,v\} \in E\}| \qquad \square$$

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The Expander-Mixing Lemma

Lemma (Haemers, 1979; Alon–Chung, 1988)

Let G be a d-regular graph on n-vertices with eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then, for any subset $U \subseteq V$ of vertices,

$$\frac{\lambda_n}{2}\left(|U| - \frac{|U|^2}{n}\right) \leq e(U) - \frac{d\,|U|^2}{2n} \leq \frac{\lambda_2}{2}\left(|U| - \frac{|U|^2}{n}\right).$$

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- Provided λ_2, λ_n are small, $e(U) \approx \frac{d}{n}\binom{|U|}{2}$
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- Provided λ_2, λ_n are small, $e(U) \approx \frac{d}{n}\binom{|U|}{2}$
  - Like a random graph of density $\frac{d}{n}$
- Moral: control of eigenvalues $\Rightarrow$ control of edge distribution

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Corollary (Hoffman's bound)

Given a d-regular graph G, we have

$$\alpha(G) \leq \left(\frac{-\lambda_n}{d - \lambda_n}\right) n.$$

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$\Rightarrow 0 \geq (d - \lambda_n)|U| + \lambda_n n$

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Expander-Mixing Proof

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Recall

$$\alpha_1 = \frac{|U|}{\sqrt{n}} \quad \text{and} \quad \sum_i \alpha_i^2 = |U|$$

Proof (ctd).

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Expand in the eigenvector basis

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$$\Rightarrow e(U) - \frac{d|U|^2}{2n} = \frac{1}{2} \sum_{i \geq 2} \lambda_i \alpha_i^2$$

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- Upper bound similar
  - ▶ Bound $\lambda_i \leq \lambda_2$ for all $i \geq 2$ $\square$

§4 Proving Erdős–Ko–Rado

I. The Classics

Erdős–Ko–Rado: Structure & Sparsification

Piecing it all together

Theorem (Erdős–Ko–Rado, 1961)
For all $n \geq 2k$, $\alpha(KG(n,k)) = \binom{n-1}{k-1}$.

Piecing it all together

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For all $n \geq 2k$, $\alpha\left(KG(n,k)\right) = \binom{n-1}{k-1}$.

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Given a d-regular graph G, we have

$$\alpha(G) \leq \left(\frac{-\lambda_n}{d - \lambda_n}\right) n.$$

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The missing link
What are the eigenvalues of $KG(n,k)$?

Lovász to the rescue

Theorem (Lovász, 1979)

Let $n \geq 2k$, and consider the Kneser graph $KG(n,k)$. The distinct eigenvalues of the adjacency matrix are, for $0 \leq j \leq k$,

$$\eta_j := (-1)^j \binom{n-k-j}{k-j}.$$

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Moreover, η_j appears with multiplicity $\binom{n}{j} - \binom{n}{j-1}$ if $j \geq 1$, and with multiplicity 1 if $j = 0$.

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Sanity check

- Size of eigenvalues
  - $\eta_0 \geq \eta_2 \geq \eta_4 \geq \ldots \geq \eta_5 \geq \eta_3 \geq \eta_1$

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Sanity check

- Size of eigenvalues
  - $\eta_0 \geq \eta_2 \geq \eta_4 \geq \ldots \geq \eta_5 \geq \eta_3 \geq \eta_1$
- Sum of multiplicities
  - $1 + \sum_{j=1}^{k} \left(\binom{n}{j} - \binom{n}{j-1} \right) = 1 + \binom{n}{k} - \binom{n}{0} = \binom{n}{k}$

Deducing Erdős–Ko–Rado

Theorem (Erdős–Ko–Rado, 1961)

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Plug into Hoffman's bound:

- $\alpha(KG(n,k)) \leq \frac{-\lambda_N}{d-\lambda_N} \cdot N = \frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$ □

Kneser's eigenvalues

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Dilemma

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- Don't know each other well enough yet for you to trust me

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- Don't have time to cover the proof of this in lecture
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Kneser's eigenvalues

Theorem (Lovász, 1979)

The distinct eigenvalues of $KG(n, k)$ are, for $0 \leq j \leq k$,

$$\eta_j := (-1)^j \binom{n-k-j}{k-j},$$

with multiplicity $\binom{n}{j} - \binom{n}{j-1}$.

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- To convince you, will highlight some important eigenvectors

A proof by example

$\eta_0 = \binom{n-k}{k}$

This is the degree of $KG(n,k)$

$\Rightarrow \vec{1}$ is an eigenvector

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We will construct eigenvectors corresponding to the stars

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Eigenspaces should be orthogonal

- Will need to remove the constant component

A proof by example (II)

$\eta_1 = -\binom{n-k-1}{k-1}$

Recall: for $i \in [n]$, $\mathcal{F}_i = \left\{ F \in \binom{[n]}{k} : i \in F \right\}$

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$$\Rightarrow \vec{f}_i(S) = \vec{g}_i(S) - \frac{k}{n} = \begin{cases} 1 - \frac{k}{n} & \text{if } i \in S \\ \frac{-k}{n} & \text{if } i \notin S \end{cases}$$

A proof by example (III)

Goal

To show $\left(A\vec{f_i}\right)(S) = -\binom{n-k-1}{k-1}\vec{f_i}(S)$ for all $S \in \binom{[n]}{k}$

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$= -\binom{n-k-1}{k-1}\left(1 - \frac{k}{n}\right) = -\binom{n-k-1}{k-1}\vec{f_i}(S)$

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$$= -\binom{n-k-1}{k-1}\vec{f_i}(S)$$

So $\vec{f_i}$ is indeed an eigenvector with eigenvalue $-\binom{n-k-1}{k-1}$

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Counting η_1-eigenvectors

We constructed n eigenvectors, $\vec{f}_i$ for $i \in [n]$

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$$\sum_{i \in [n]} \vec{f_i}(S) = \sum_{i \in S} \vec{f_i}(S) + \sum_{i \notin S} \vec{f_i}(S)$$
$$= k\left(1 - \frac{k}{n}\right) + (n-k)\left(\frac{-k}{n}\right) = 0 \qquad \square$$

An eigensummary

What we now hopefully believe

$\eta_0 = \binom{n-k}{k}$ and $\eta_1 = -\binom{n-k-1}{k-1}$ are indeed eigenvalues

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Can construct $\binom{n}{k}$ eigenvectors $\Rightarrow$ we have them all

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$\Rightarrow$, e.g., that the span of the η_0 and η_1 eigenspaces is generated by characteristic vectors $\vec{g}_i$ of stars

II. Spectral Stability

Erdős–Ko–Rado: Structure & Sparsification
8th Polish Combinatorial Conference

Chapter outline

1. Uniqueness in Erdős–Ko–Rado

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3. The Spectral Approach

§1 Uniqueness in Erdős–Ko–Rado

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Missing characterisation

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Theorem (Erdős–Ko–Rado, 1961)

For all $n \geq 2k$, if $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.
Moreover, if $n > 2k$, then the only maximum intersecting families are the stars.

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Can we get the same result with spectral techniques?

A spectral characterisation

Equality in Hoffman's bound

- Only one single inequality in the proof
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- λ_1 is η_0 and λ_n is η_1
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- Can we show it must actually be such a vector?

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- $\Rightarrow$ we are interested in Boolean functions $f : \{0,1\}^n_k \to \{0,1\}$

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Corollary

The span of the η_0- and η_1-eigenspaces is precisely the set of affine functions $g : \{0,1\}^n_k \to \mathbb{R}$; that is, functions of the form $g(\vec{x}) = c_0 + \sum_{i=1}^n c_i x_i$, for some constants $c_0, c_1, \ldots, c_n$.

Boolean dictatorships

Lemma (Filmus, 2016)

Let $2 \leq k \leq n-2$, *and suppose* $g : \{0,1\}^n_k \to \{0,1\}$ *is affine. Then either* g *is constant, or there is some* $i \in [n]$ *such that* $g(\vec{x}) = x_i$ *or* $g(\vec{x}) = 1 - x_i$.

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- Then $g(S) - g(S') = c_j - c_1$
- Since g is a Boolean function (and $c_j \geq c_1$), $c_j - c_1 \in \{0,1\}$
- $\Rightarrow c_j \in \{c_1, c_1 + 1\}$ for all j

Boolean dictatorships (II)

Case: all c_j equal c_1

- Then $g(\vec{x}) = c_0 + c_1 \sum_i x_i = c_0 + kc_1$
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This covers all possible cases, and so the lemma is proven $\square$

Completing Erdős–Ko–Rado

Corollary

If $n > 2k$, $\mathcal{F} \subseteq \binom{[n]}{k}$ *is intersecting, and* $|\mathcal{F}| = \binom{n-1}{k-1}$, *then* $\mathcal{F}$ *is a star.*

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$g = 0 \Leftrightarrow \mathcal{F} = \emptyset$ — ignore

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$\Rightarrow$ only left with the case $g = x_i$, which is a star $\square$

§2 Stability Results

II. Spectral Stability
Erdős–Ko–Rado: Structure & Sparsification

A question of stability

The extremal question

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Nontrivial families

- Natural question: how large can a nontrivial intersecting family be?
- Describe the structure of large intersecting families

The Hilton–Milner Theorem

Theorem (Hilton–Milner, 1967)

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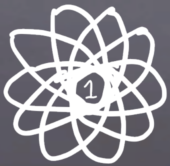

A full star

A Hilton–Milner family

More on Hilton–Milner

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- Other measures of nontriviality?

More stability measures

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Diversity (Frankl, 1980; Kupavskii, 2017)

The diversity of an intersecting family is the minimum number of sets that must be removed to make the family trivial. It has been shown that the larger the diversity, the smaller the family.

§3 The Spectral Approach

II. Spectral Stability

Erdős–Ko–Rado: Structure & Sparsification

Objectives

Large families

- We focus on intersecting families of size near $\binom{n-1}{k-1}$
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- Previously: bounded λ_i from below by λ_n, solved for $|U|$
- Now: quantify the error when estimating

Keeping an eye on errors

Idea

Instead of simply replacing λ_i with λ_n, we add a $\lambda_i - \lambda_n$ correction term separately

Edge distribution

We have

$$\begin{aligned} e(U) &= \frac{d\,|U|^2}{2n} + \frac{1}{2}\sum_{i\geq 2}\lambda_i\alpha_i^2 \\ &= \frac{d\,|U|^2}{2n} + \frac{1}{2}\lambda_n\sum_{i\geq 2}\alpha_i^2 + \frac{1}{2}\sum_{i\geq 2}\left(\lambda_i - \lambda_n\right)\alpha_i^2 \\ &= \frac{d\,|U|^2}{2n} + \frac{1}{2}\lambda_n\left(|U| - \frac{|U|^2}{n}\right) + \frac{1}{2}\sum_{i\geq 2}\left(\lambda_i - \lambda_n\right)\alpha_i^2 \end{aligned}$$

A later approximation

Recall

$$e(U) = \frac{d\,|U|^2}{2n} + \frac{1}{2}\lambda_n \left(|U| - \frac{|U|^2}{n} \right) + \frac{1}{2}\sum_{i \geq 2} (\lambda_i - \lambda_n)\,\alpha_i^2$$

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- $\Rightarrow$ error comes from non-affine part of the characteristic vector of U

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- Calculations can all be carried out explicitly, straightforward if tedious

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A hopeful query

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    - Only $\binom{n-2}{k-2}$ such sets, very small compared to $\binom{n}{k}$
    - Does not cause great difference in the ℓ_2 norm

Nearly-affine functions

A hopeful query

- Previous lemma: affine Boolean function $\leftrightarrow$ star
- Nearly-affine function $\leftrightarrow$ a family close to a star?

A disappointing, if vigorous, answer

- No!
- Consider $f(\vec{x}) = \max\{x_1, x_2\}$ — characteristic function of union of stars with centres 1 and 2
- Close to the affine function $h(\vec{x}) = x_1 + x_2$
  - Only differ when $x_1 = x_2 = 1$, i.e. the intersection of the stars
  - Only $\binom{n-2}{k-2}$ such sets, very small compared to $\binom{n}{k}$
  - Does not cause great difference in the ℓ_2 norm
- However, a union of two stars is twice the size of one star — very different in structure

Filmus to the rescue!

Affine stability

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Theorem (Filmus, 2018)

If a Boolean function $f : \{0,1\}^n_k \to \{0,1\}$ is close to an affine function, then there is some set $S \subseteq [n]$ of bounded size such that either f or $1 - f$ is close to $\max_{i \in S} x_i$.

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Proof ideas

- Uses the famous Friedgut–Kalai–Naor Theorem on nearly-affine Boolean functions on the Hamming cube $\{0,1\}^n$
- Applies this theorem to random subcubes within $\{0,1\}_d^n$
- Analyses random output to reach the conclusion

Putting it all together

Recall

$$e(U) = \frac{d\,|U|^2}{2n} + \frac{1}{2}\lambda_n \left(|U| - \frac{|U|^2}{n} \right) + \frac{1}{2} \sum_{i:\lambda_i \neq \lambda_1, \lambda_n} (\lambda_i - \lambda_n)\, \alpha_i^2$$

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- Filmus $\Rightarrow$ close to a union of stars
- Since size is $\approx \binom{n-1}{k-1}$, must be a single star

Further robustness

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- $\Rightarrow$ obtain stability for large families with relatively few disjoint pairs

The final outcome

Theorem (D.–Tran, 2016)

There is an absolute constant $C > 1$ *such that if* n *and* k *are positive integers satisfying* $n > 2k$, *and* $\mathcal{F} \subset \binom{[n]}{k}$ *is a family of size* $|\mathcal{F}| = (1-\alpha)\binom{n-1}{k-1}$ *with at most* $\beta\binom{n-1}{k-1}\binom{n-k-1}{k-1}$ *disjoint pairs, where* $\max(2|\alpha|, |\beta|) \leq \frac{n-2k}{(20C)^2 n}$, *then there is a star* $\mathcal{S}$ *satisfying* $|\mathcal{F}\Delta\mathcal{S}| \leq C(\alpha + 2\beta)\frac{n}{n-2k}\binom{n-1}{k-1}$.

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- Friedgut and Regev proved a more general, but less quantitative, removal lemma

Dziękuję